

QUOTIENTS OF K3 SURFACES MODULO INVOLUTIONS

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ABSTRACT. Let X be a K3 surface with an involution σ which has non-empty fixed locus X^σ and acts non-trivially on a non-zero holomorphic 2-form. We shall construct all such pairs (X, σ) in a canonical way, from some better known double coverings of log del Pezzo surfaces of index ≤ 2 or rational elliptic surfaces, and construct the only family of each of the three extremal case where X^σ contains 10 (maximum possible) curves. We also classify rational log Enriques surfaces of index 2.

Introduction

Let X be a smooth projective K3 surface over the complex number field \mathbf{C} . Let σ be an involution on X . This σ induces an action on a non-zero holomorphic 2-form ω of X such that $\sigma^*\omega = \pm\omega$. If $\sigma^*\omega = \omega$ then the quotient space X/σ is again a K3 surface with at worst type A_1 Du Val singular points.

We are interested in the case where $\sigma^*\omega = -\omega$. We want to determine the structure of $S := X/\sigma$ and construct all such double coverings $\pi : X \rightarrow S$, as canonical resolutions [H1, p. 48], of some better known double coverings $\pi_c : X_c \rightarrow S_c$ (Theorem 1). S_c is, except for the Enriques case (Lemma 1.2(3)) and the log del Pezzo case (Lemma 3.2), a rational elliptic surface and π_c ramifies over one or two fibers.

We refer to [R] for the comparison of advantages of two approaches : top-down and bottom-up. For involution-actions on the K3 lattice, we refer to [N1,2].

In this paper we shall prove the following Theorems 1, 3 and 4 and Corollary 5. Theorem 1 below, in the cases of Lemmas 2.1 and 3.2, also tells the relations between rational log Enriques surfaces and log del Pezzo surfaces downstairs, which have been studied in [AN, Bl, D, MZ1,2, N3,4,5, Z1,2,3], and K3 surfaces with an involution upstairs.

Theorem 1. *Let X be a smooth projective K3 surface with an involution σ such that $\sigma^*\omega = -\omega$ for a non-zero holomorphic 2-form ω . Let $\pi : X \rightarrow S := X/\sigma$ be the quotient morphism. Then the following two assertions hold.*

- (1) *The fixed locus X^σ is empty if and only if S is an Enriques surface.*
- (2) *Suppose that $X^\sigma \neq \emptyset$. Then π is the canonical resolution, in the sense of Horikawa [H1, p.48], of a double covering $\pi_c : X_c \rightarrow S_c$ so that π and π_c are precisely constructed in Lemma 2.4, 3.2, 4.1, 5.1 or 6.1. In the cases of Lemmas 4.1 and 5.1, we have $\pi = \pi_c$.*

In particular, one has the following commutative diagram with f as the minimal resolution and g a resolution

$$\begin{array}{ccc} X & \xrightarrow{f} & X_c \\ \pi \downarrow & & \downarrow \pi_c \\ S & \xrightarrow{g} & S_c. \end{array}$$

Remark 2. (1) We call $\pi_c : X_c \rightarrow S_c$ a *canonical mapping model* of $\pi : X \rightarrow S$. This model is unique in the cases of Lemmas 3.2, 4.1, 5.1 and 6.1, but not unique in the case of Lemma 2.4 (see Definition and Propositions 2.10, 3.5 and 6.4).

(2) This theorem also shows the usefulness of Persson's list given in [P], because in all cases, except for the case of Lemma 3.2, the canonical mapping model is constructed from a rational relatively minimal elliptic fibration.

Theorem 3 (see Theorem 3' at the end of §1 for the detailed version). *Let (X, σ) be the pair satisfying the hypothesis of Theorem 1. Let X^σ be the fixed locus. Then the following four assertions are true.*

(1) X^σ is a disjoint union of m smooth curves for some $0 \leq m \leq 10$ and m can attain any value in this range. If $m = 10$, then (X, σ) has one of the following three types.

Type(Rat). X^σ is a union of 10 rational curves.

Type(Gn2). X^σ is a union of one genus-two curve and 9 rational curves.

Type(Ell). X^σ is a union of one elliptic curve and 9 rational curves.

(2) There is, up to isomorphisms, only one pair (X, σ) of Type(Rat).

(3) There is a family $\mathcal{X}_{\text{ell}} := \{(X_s, \sigma_s) | s \in \mathbf{P}^1\}$ of surface X_s and an involution σ_s on it such that for $s \neq \infty, 0, 1, s_0$, the pair (X_s, σ_s) is of Type(Ell). Conversely, every pair of Type(Ell) is isomorphic to (X_s, σ_s) for some $s \neq \infty, 0, 1, s_0$.

(4) There is a family $\mathcal{Y}_{\text{gn2}} = \{(Y_t, \sigma_t) | t \in \mathbf{P}^3\}$ of surface Y_t and an involution σ_t on it such that for a general t , the pair (Y_t, σ_t) is of Type(Gn2). Conversely, every pair of Type(Gn2) is isomorphic to (Y_t, σ_t) for some t .

Remark. (1) Nikulin [N2] classified the configuration of X^σ in the case $\sigma^*|_{\text{Pic}X} = \text{id}$. However, our argument and result are more geometrical, even in this case.

(2) We also prove in Theorem 3' that the Picard number $\rho(X) \geq 18$ in all three extremal cases. We refer to [Mo] for K3 surfaces with large Picard number.

(3) Though we constructed a 3-dimensional family \mathcal{Y}_{gn2} of K3 surfaces of Type(Gn2) and Picard number ≥ 18 , we have not identified K3 surfaces in the family which are isomorphic to each other, and not all, I guess, of K3 surfaces of Picard number ≥ 18 are included in this family. Therefore, this dimension 3 may not give any restriction on the dimensions of moduli spaces of K3 surfaces with Picard number ≥ 18 .

In view of Theorem 3' (3)(4), the extremal Type(Rat) is also, with one or two smooth rational curves, contracted, the degeneration of other two extremal Types (Ell) and (Gn2). So *the most extremal pair* should be a right name for the pair of Type(Rat) (see [OZ1,2]). Also, Theorem 3'(4) supports the naming of extremal case in [AN] for the surface S .

Our next Theorem 4 reduces, in the sense of the 3-column diagram below, the classification of rational log Enriques surfaces of index 2 to that of pairs (X, σ) as in Theorem 1: the case of Lemma 2.4, and hence reduce further to that of double coverings of rational elliptic surfaces ramifying over one or two fibers (Lemma 2.4).

Theorem 4. *Let R be a rational log Enriques surface of index 2 with $\bar{\pi} : W \rightarrow R$ as its canonical covering (Definition 1.7).*

Then there exists a (smooth) K3 surface X with an involution σ such that the fixed locus X^σ is a disjoint union of n ($1 \leq n \leq 10$) smooth rational curves, and that the quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution of $\bar{\pi}$ (Definition 2.2).

In particular, one has the following commutative diagram, where $\pi_c : X_c \rightarrow S_c$ is a canonical mapping model of π (Definition 2.10), and where p, f are the minimal resolutions and q, g resolutions all precisely described in Lemmas 2.1 and 2.4

$$\begin{array}{ccccc} W & \xleftarrow{p} & X & \xrightarrow{f} & X_c \\ \bar{\pi} \downarrow & & \pi \downarrow & & \downarrow \pi_c \\ R & \xleftarrow{q} & S & \xrightarrow{g} & S_c. \end{array}$$

Corollary 5. *Let R be a rational log Enriques surface of index 2 and let r be the number of singular points of Cartier index 2. Then the following two assertions are true.*

- (1) *One has $1 \leq r \leq 10$, and r can attain any value in this range.*
- (2) *There is, up to isomorphisms, only one rational log Enriques surface of index 2 with $r = 10$ (see Example 2.8 for the construction of this surface).*

Remark 6. Let R be the unique rational log Enriques surface of Type A_{19} constructed in [Z2, Example 3.2 and Theorem 3.6] (see [OZ1, Example 2] and Example 2.8 below for two more different constructions, of the same surface). The uniqueness theorem [OZ1, Theorem 2] is the answer to the question asked by Naruki, and Reid who also discussed it in [R, Example 6]. Let $\bar{\pi} : W \rightarrow R$ be the canonical double covering (Definition 1.7) and let $\pi : X \rightarrow S = X/\sigma$ be the canonical resolution of $\bar{\pi}$ (Definition 2.2).

Then the pair (X, σ) here is isomorphic to the pair of Type(Rat) in Theorem 3, while the surface obtained from S by contracting $\pi(X^\sigma)$ into 10 cyclic-quotient singular points of Brieskorn type $C_{4,1}$, is isomorphic to the unique rational log Enriques surface of index 2 in Corollary 5(2) with $r = 10$ (see proofs of Theorem 3 and Corollary 5 and Example 2.8).

The organisation of the paper is as follows. The first section is on how the fixed locus X^σ sits in the surface X . Lemmas 1.5, 1.6 and 1.11, which might be of general interest, describe how σ acts on elliptic fibers and Dynkin diagrams.

§2 ~ §6 form the main ingredients used in §7 to prove the theorems. All three extremal pairs (X, σ) in Theorem 3 or Theorem 3' are precisely constructed in Example 2.8 and §7 both by taking as S_c (see Theorem 1 for the notation) the

same smooth rational surface with a multiple-fiber free elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$ having $\{I_9, 3I_0\}$ as its set of singular fibers. Such a pair (S_c, ψ) is unique up to isomorphisms (Lemma 7 in §7). This S_c or the pair (S_c, ψ) will be called *Persson's most extremal rational elliptic surface*.

As an application to [OZ1, Theorem 2], we show in Example 2.8 the following:

Let S_{ci} ($i = 1, 2$) be a smooth rational surface with a multiple-fiber free elliptic fibration $\psi_i : S_{ci} \rightarrow \mathbf{P}^1$ which has two singular fibers $\overline{F}_{i,j}$ ($j = 1, 2$) of Kodaira type II (setting $n_{i,j} := 1$), or III ($n_{i,j} := 2$), or IV ($n_{i,j} := 4$), or $I_{n_{i,j}}$ ($n_{i,j} \geq 1$) such that $n_{i,1} + n_{i,2} = 10$. For instance, one can take Persson's pair as (S_{ci}, ψ_i) . Let $g_i : S_i \rightarrow S_{ci}$ be a smooth blowing-up of points in $\overline{F}_{i,j}$ so that $F_{i,j} := g_i^* \overline{F}_{i,j}$ fits respectively Case (α) , or (β) , or (γ) , or (δ) in Lemma 1.5. Then one has $S_1 \cong S_2 \cong X/\sigma$ where (X, σ) is the unique pair of Type(Rat) in Theorem 3.

The above result suggests that one can divide Persson's list in [P] of rational surfaces with an elliptic fibration into several classes so that any two members in the same class can be transformed to each other by a blowing-up succeeded by a blowing-down both similar to the g_i above.

In §7, we prove, as a corollary to Lemma 7, that there is, up to isomorphisms, only one log del Pezzo surface with a type A_8 Du Val singular point as its only singular point.

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§1. Preliminaries

The following Lemma 1.1 is also proved in [OZ1].

Lemma 1.1. *Let (X, σ) be the same as in Theorem 1. Then the following three assertions hold.*

- (1) *The fixed locus X^σ is a disjoint union of smooth curves.*
- (2) *Let G_1 and G_2 be two σ -stable smooth rational curves with $G_1 \cdot G_2 = 1$. Then exactly one of G_i is σ -fixed.*
- (3) *Let G be a σ -stable but not σ -fixed smooth rational curve. Then $\#(G \cap X^\sigma) = 2$.*

Proof. (1) By the hypothesis on σ , at any σ -fixed point P , we have $\sigma(x, y) = (x, -y)$ for suitable local coordinates at P . So P lies on the σ -fixed curve $y = 0$, which is smooth.

(3) follows from Hurwitz's genus formula applied to $\pi : G \rightarrow G/\pi$.

(2) Since G_1, G_2 are σ -stable, the intersection $P := G_1 \cap G_2$ is σ -fixed, i.e., $P \in X^\sigma$. Note that a generator of the tangent space to G_i at P , is an eigenvector of σ_* with respect to an eigenvalue λ_i . By the conditions on σ , one has $\{\lambda_1, \lambda_2\} = \{1, -1\}$ as sets. That is, for exactly one $i \in \{1, 2\}$, one has $\lambda_i = 1$ and G_i is σ -fixed. This proves Lemma 1.1.

Lemma 1.2. *Let (X, σ) be as in Theorem 1. Let $S := X/\sigma$ be the quotient space and $\pi : X \rightarrow S$ the quotient morphism. Then the following four assertions hold true.*

- (1) *S is a smooth surface with irregularity $q(S) = 0$.*
- (2) *Let $X^\sigma = \coprod_{i=1}^m C_i$, where C_i is a smooth irreducible curve. Then $\pi^* D_i = 2C_i$ where $D_i := \pi(C_i)$, $0 \sim \pi^* K_S + \sum_{i=1}^m C_i$ and $-2K_S \sim \sum_{i=1}^m D_i$.*
The quotient morphism $\pi : X \rightarrow S$ coincides with the double covering $\text{Spec } \oplus_{i=0}^1 \mathcal{O}(K_S)^{\otimes 2} \rightarrow S$ associated with the relation $\mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}(\sum_{i=1}^m D_i)$.
- (3) *Suppose that $X^\sigma = \phi$. Then S is an Enriques surface and $\pi : X \rightarrow S$ is the canonical unramified covering associated with the relation $\mathcal{O}(K_S)^{\otimes 2} \cong \mathcal{O}_S$.*
- (4) *Suppose that $X^\sigma \neq \phi$. Then S is a rational surface.*

Proof. (1) follows from Lemma 1.1 (1) and the fact that $q(S) \leq q(X) = 0$.

(2) By the ramification formula $0 \sim K_X = \pi^* K_S + \sum_{i=1}^m C_i$. This, together with the projection formula, implies $-2K_S \sim \sum_{i=1}^m D_i$. Now the second paragraph of (2) follows by applying [H2, Lemma 2.1; H1, Lemma 4].

(3) By the assumption on S , we see that K_S is not linearly equivalent to zero. Now (3) follows from (2) and (1).

(4) Since $-2K_S \sim \sum_{i=1}^m D_i > 0$, the pluri-genera $P_n(S) = 0$ for all $n \geq 1$. Hence S is rational because $q(S) = 0$. This completes the proof of Lemma 1.2.

Definition 1.3. Let T be a normal projective surface with at worst quotient singular points. For any $t \in \text{Sing}T$, there exists a “small” finite group $G_t \subseteq GL_2(\mathbf{C})$ such that (T, t) is isomorphic to $(\mathbf{C}^2/G_t, 0)$ analytically. The index $|G_t : G_t \cap SL_2(\mathbf{C})|$ is called *the Cartier index* of t .

Note that $t \in \text{Sing}T$ has Cartier index 1 if and only if t is a Du Val singular point.

The Cartier index of a smooth point is defined as 1. *The Cartier index* $I = I(T)$ of T is defined as the l.c.m. of Cartier indices of all points on T . Note that I is nothing but the smallest positive integer such that the multiple IK_T of the canonical divisor K_T is a Cartier divisor.

Lemma 1.4. *Let T be a normal projective surface with at worst quotient singular points of index ≤ 2 , and let t_i ($i = 1, 2, \dots, r; r \geq 0$) be all singular points on T of index 2. Let $g_1 : S_1 \rightarrow T$ be the minimal resolution. Then the following three assertions are true.*

- (1) *t_i is cyclic of Brieskorn type $C_{4n_i, 2n_i-1}$ ($n_i \geq 1$). Hence $g_1^{-1}(t_i)$ is either a single (-4) -curve ($n_i = 1$), or a linear chain of two (-3) -curves as tips and $n_i - 2$ ($n_i \geq 2$) (-2) -curves as middle components:*

$$(-4), \quad (-3) - (-2) - (-2) - \dots - (-2) - (-2) - (-3).$$

- (2) *Let $g_2 : S \rightarrow S_1$ be the smooth blowing-up of all intersection points of $g_1^{-1}(t_i)$ for those i with $n_i \geq 2$. Let $g = g_1 \circ g_2 : S \rightarrow T$. Then $g^{-1}(t_i)$ is either a single (-4) -curve $D_{i,1}$ ($n_i = 1$), or a linear chain of n_i (-4) -curves $D_{i,j}$ ($j = 1, 2, \dots, n_i; n_i \geq 2$) and $n_i - 1$ (-1) -curves $H_{i,j}$ ($j = 1, 2, \dots, n_i - 1$):*

$$D_{i,1} \quad D_{i,2} \quad \dots \quad D_{i,n_i} \quad H_{i,1} \quad H_{i,2} \quad \dots \quad H_{i,n_i-1}$$

$$(3) K_S = g^* K_T - 1/2 \sum_{i=1}^r \sum_{j=1}^{n_i} D_{i,j}.$$

Proof. Let $B := g_1^{-1}(\text{Sing} T) = \sum_k B_k$ be the irreducible decomposition. By the equivalence of quotient singularity (resp. Du Val singularity) and log terminal singularity (resp. canonical singularity) [Ka1, Cor. 1.9] (see also [KMM] and [Ko]), there are rational numbers α_i with $0 \leq \alpha_i < 1$ such that

$$K_{S_1} = g_1^* K_T - \sum_k \alpha_k B_k.$$

Moreover, $\alpha_k = 0$ if and only g_1 maps the connected component of B containing B_k into a Du Val singular point.

Since $2K_T$ is Cartier by the hypothesis, $2\alpha_k$ is an integer. This, together with $0 \leq \alpha_k < 1$, implies that $\alpha_k = 0$ or $1/2$.

Now the second assertion in (1) is the consequence of the calculation of the conditions $(K_{S_1} + \sum_k \alpha_k B_k) \cdot B_t = 0$ (cf. the proof of [Z2, Lemma 1.8]), whence the first assertion in (1) also follows.

(2) follows from (1) and the construction of g_2 , while (3) follows from the above ramification formula involving g_1 . This proves Lemma 1.4.

Lemma 1.5. *Let X, σ and $\pi : X \rightarrow S := X/\sigma$ be as in Theorem 1. Suppose that there exists an elliptic fibration $\varphi : X \rightarrow \mathbf{P}^1$ such that X^σ is contained in fibers. Let E_i be a σ -stable singular fiber. Then $\text{Supp} E_i = \sum_{j=1}^{n_i} C_{i,j} + \sum_j G_{i,j}$ is one of the following five cases, where $C_{i,j}$'s are only σ -fixed components contained in E_i .*

Case(A). E_i is of Kodaira type IV. So E_i is a union of three curves $C_{i,1}, G_{i,1}, G_{i,2}$ which share the same point. One has $\sigma(G_{i,1}) = G_{i,2}$. ■

Case(B). E_i is of Kodaira type I_0^* . Hence E_i is a tree with a central component $G_{i,1}$ and four curves $G_{i,2}, G_{i,3}, C_{i,1}, C_{i,2}$ sprouting out from $G_{i,1}$. This $G_{i,1}$ is σ -stable while $\sigma(G_{i,2}) = G_{i,3}$.

Case(C). E_i is of Kodaira type IV^* . So E_i is a tree with a central component $C_{i,4}$ and three twigs $G_{i,j} + C_{i,j}$ ($j = 1, 2, 3$) sprouting out from $C_{i,4}$ such that $G_{i,j} \cdot C_{i,4} = 1$. Each $G_{i,j}$ is σ -stable.

Case(D). E_i is of Kodaira type I_{2n_i} . Hence $E_i = C_{i,1} + G_{i,1} + C_{i,2} + G_{i,2} + \cdots + C_{i,n_i} + G_{i,n_i}$ is a loop so that $C_{i,j} \cdot G_{i,j} = G_{i,j} \cdot C_{i,j+1} = 1$. Here we set $C_{i,n_i+1} = C_{i,1}, G_{i,n_i+1} = G_{i,1}$. Each $G_{i,j}$ is σ -stable.

Case(E). E_i is of Kodaira type I_{2s_i} ($s_i \geq 1$). So $E_i = \sum_{j=1}^{2s_i} G_{i,j}$ is a loop so that $G_{i,j} \cdot G_{i,j+1} = 1$. Here we set $G_{i,2s_i+1} = G_{i,1}$. One has $\sigma(G_{i,j}) = G_{i,s_i+j}$.

Finally, $\pi_* E_i$ fits one of the following cases according to the type of E_i . Here $\pi^* D_{i,j} = 2C_{i,j}$ and $\sum_{j=1}^{n_i} D_{i,j}$ is a disjoint union of n_i (-4) -curves $D_{i,j}$. One has also $\pi^* H_{i,j} = G_{i,j}$, except for Case(α) where $\pi^* H_{i,1} = G_{i,1} + G_{i,2}$, Case(β) where $\pi^* H_{i,2} = G_{i,2} + G_{i,3}$ and Case(ε) where $\pi^* H_{i,j} = G_{i,j} + G_{i,s_i+j}$.

Case(α). $F_i := \pi_* E_i = 2H_{i,1} + D_{i,1}$ is a union of the touching (-1) -curve $H_{i,1}$ and (-4) -curve $D_{i,1}$.

Case(β). $F_i := \pi_* E_i = 4H_{i,1} + 2H_{i,2} + D_{i,1} + D_{i,2}$ is a tree. $H_{i,1}$ is a (-1) -curve and also the central component which meets the (-2) -curve $H_{i,2}$ and the (-4) -curves $D_{i,1}, D_{i,2}$.

Case(γ). $F_i := \pi_* E_i = 3D_{i,4} + 4H_{i,1} + D_{i,1} + 4H_{i,2} + D_{i,2} + 4H_{i,3} + D_{i,3}$ is a tree. $D_{i,4}$ is a (-4) -curve and also the central component meeting three (-1) -curves

$H_{i,1}, H_{i,2}, H_{i,3}$, while the (-4) -curve $D_{i,j}$ ($j = 1, 2, 3$) is a tip component meeting only $H_{i,j}$.

Case(δ). $F_i := \pi_* E_i = D_{i,1} + 2H_{i,1} + D_{i,2} + 2H_{i,2} + \cdots + D_{i,n_i} + 2H_{i,n_i}$ is a simple loop with $D_{i,j} \cdot H_{i,j} = H_{i,j} \cdot D_{i,j+1} = 1$. Here $D_{i,j}$ is a (-4) -curve while $H_{i,j}$ is a (-1) -curve.

Case(ε). $\pi_* E_i = 2F_i$, where $F_i = \sum_{j=1}^{s_i} H_{i,j}$ is of Kodaira type I_{s_i} .

Proof. Set $E := E_i$. By Lemma 1.1 and the hypothesis on φ , one obtains:

Claim(1). (1.1) If P is a point in $E \cap X^\sigma$, then $E \cap X^\sigma$ contains a smooth component through P .

(1.2) Suppose that G is a σ -stable but not σ -fixed curve in E . Then E contains either one component C_1 or two components C_1, C_2 of X^σ such that either $\#(G \cap C_1) = 2$, or $\#(G \cap C_i) = 1$ for both $i = 1, 2$, accordingly.

By the classification theory, E has one of the following 10 Kodaira types:

- (A) Type IV, (B) Type I_0^* , (C) Type IV^* , (DE) Type I_m ($m \geq 2$),
- (F) E is a nodal or cuspidal rational curve,
- (G) E is a union of two touching smooth rational curves E_1, E_2 ,
- (H) Type I_n^* ($n \geq 1$), and (I) Type II^* or III^* .

If E is as in Case(F), then the singular point of E is σ -fixed. We reach a contradiction to Claim(1.1).

If E is as in Case(G), then the common point of E_1 and E_2 is σ -fixed. By Claim(1.1) and Lemma 1 (1), one may assume that E_1 is σ -fixed, while E_2 is σ -stable but not σ -fixed. This contradicts Claim(1.2) applied to $G := E_2$.

Let E be as in Case(H). So $E = \sum_{i=1}^{n+1} E_i + E_{1,1} + E_{1,2} + E_{n+1,1} + E_{n+1,2}$ consists of a linear chain

$$E_1 + E_2 + \cdots + E_n + E_{n+1}$$

and curves $E_{i,1}, E_{i,2}$ sprouting out from E_i ($i = 1$ and $n + 1$). Note that either $\sigma(E_i) = E_{n+2-i}$ for all i or E_i is σ -stable for all i .

In the first subcase, no E_i is σ -fixed. However, when $n + 1$ is odd (resp. even), the middle component $E_{(n+2)/2}$ (resp. the intersection $E_{(n+1)/2} \cap E_{(n+1)/2+1}$) is σ -stable. This contradicts Claim(1).

In the second subcase, each E_i ($i = 1, n + 1$) is not σ -fixed by applying Lemma 1.1 (1) and Claim (1.2) to $G := E_{i,1}$. Thus, at least one of $E_{i,1}, E_{i,2}$, say $E_{i,1}$, is σ -fixed by Claim(1.2) applied to $G := E_i$. Then $E_{i,2}$ is σ -stable. So E_i would have three σ -fixed points $E_i \cap E_{i,1}, E_i \cap E_{i,2}, E_i \cap E_j$ where $j = i + 1$ (resp. $i - 1$) if $i = 1$ (resp. $i = n + 1$). Hence E_i must be σ -fixed. We reach a contradiction.

Let E be as in Case(I). Then E consists of a central component R and three twigs T_i sprouting out from R . Note that R is σ -stable. But R is not σ -fixed by applying Lemma 1.1(1) and Claim(1.2) to $G :=$ (the shortest twig among T_i 's). Then, by Claim(1.2), R meets a σ -fixed curve in T_i for $i = 1$ and 2 say. Hence all three twigs T_i are σ -stable. So R would have three σ -fixed points $R \cap T_i$, whence R must be σ -fixed. This is a contradiction.

By the same arguments as above, we can prove that Lemma 1.5 is true if E is of type (A), (B), (C) or (DE).

Lemma 1.6. *Let (X, σ) be as in Theorem 1. Let Γ_i be a union of normal crossing smooth rational curves of Dynkin type A_m ($m \geq 1$), D_m ($m \geq 4$) or E_m ($m = 6, 7, 8$). Suppose that Γ_i is σ -stable and that every curve of X^σ is either contained in Γ_i or disjoint from Γ_i .*

Then Γ_i is of type A_{2n_i-1} as follows:

$$C_{i,1} - G_{i,1} - C_{i,2} - G_{i,2} - \cdots - C_{i,n_i-1} - G_{i,n_i-1} - C_{i,n_i}.$$

Here $C_{i,j}$ is σ -fixed, while $G_{i,j}$ is σ -stable but not σ -fixed.

Proof. The argument will be similar to Lemma 1.5. In particular, the arguments for Lemma 1.5 Case(I) implies that it is impossible that Γ_i is of type E_m or D_m . So Γ_i is of type A_m as follows:

$$B_1 - B_2 - \cdots - B_m.$$

As in Lemma 1.5 Case(H), Γ_i contains either a σ -stable curve or a σ -fixed point. Hence $X^\sigma \cap \Gamma_i \neq \emptyset$ (Lemma 1.1 (3)) and Γ_i contains a σ -fixed curve by the hypotheses of Lemma 1.6. Thus Γ_i is component-wisely σ -stable. Now Lemma 1.6 follows from Lemma 1.1.

Definition 1.7. Let R be a normal projective surface with at worst quotient singular points.

(1) R is a *log Enriques surface* if the irregularity $q(R) = h^1(R, \mathcal{O}_R) = 0$ and if a positive multiple mK_R of the canonical divisor K_R is linearly equivalent to zero. The index $I = I(R)$ is the smallest positive integer such that $IK_R \sim 0$, or equivalently the Cartier index in Definition 1.3 [Z2, Lemma 1.5].

(2) The surface $W := \text{Spec } \bigoplus_{i=0}^{I-1} \mathcal{O}(-iK_R)$ or the natural quotient morphism $\pi : W \rightarrow R$, associated with the relation $\mathcal{O}(K_R)^{\otimes I} \cong \mathcal{O}_R$, is called the *canonical covering* of R . This π is a Galois $\mathbf{Z}/I\mathbf{Z}$ -covering such that $W/(\mathbf{Z}/I\mathbf{Z}) = R$.

(3) In the sense of [OZ1], R is called of *Type* $\alpha A_a + \delta D_d + \varepsilon E_e$ if the canonical covering W satisfies $\text{Sing } W = \alpha A_a + \delta D_d + \varepsilon E_e$.

Remark 1.8. (1) Note that W has at worst Du Val singular points and $K_W \sim 0$. So W is either an abelian surface or a K3 surface with at worst Du Val singular points. Moreover, $\pi : W \rightarrow R$ is unramified over $R - \text{Sing } R$.

(2) [Z2] classified the case where W is smooth. The remaining cases of log Enriques surfaces R were dealt with in [Z3]; we proved there that there exists a crepant blowing up $Q \rightarrow R$ with a new log Enriques surface Q of the same index such that the canonical cover V of Q has at worst type A_1 Du Val singular points. $\text{Sing } Q$ and $\rho(Q)$ are tabulated there.

(3) Blache [Bl] considers normal projective surfaces with at worst log canonical singular points. He also improves the upper bound of I to that $I \leq 21$. Examples of log Enriques surfaces of all possible prime indices are given in [Z3].

(4) Recently, we proved in [OZ1, 2] that there is, upto isomorphisms, only one (resp. one or two) log Enriques surface(s) R of Type A_{19} , or D_{19} , or D_{18} (resp. A_{18}). In the first case (resp. the last three cases), the minimal resolution X of the canonical covering W of R , is the unique smooth K3 surface with discriminant of $PicX$ equal to 4 (resp. 3).

Definition 1.9. Let T be a normal projective surface with at worst quotient singular points.

(1) T is a *log del Pezzo surface* if the anti-canonical divisor $-K_T$ is a \mathbf{Q} -ample divisor.

(2) A *Gorenstein log del Pezzo surface* is a log del Pezzo surface of Cartier index 1, equivalently, a log del Pezzo surface with at worst Du Val singular points.

Remark 1.10. (1) In view of [S], log del Pezzo surfaces are rational (see also [Z4, Lemma 1.1] or [Z5, Lemma 1.3]).

(2) Alexeev and Nikulin [AN] classified log del Pezzo surfaces of Cartier index 2.

(3) [N3,4,5] gave upper bounds for the Picard number of the minimal resolution of a log del Pezzo surface T in terms of the Cartier index or the l.c.m. of multiplicities of T .

(4) S. Keel and J. McKernan have announced in the 1995 Summer School of Algebraic Geometry their affirmative answer to a conjecture of Miyanishi, which says that the smooth part $T - SingT$ of a log del Pezzo surface T is rationally connected.

(5) In [MZ 1, 2], Gorenstein log del Pezzo surfaces T are classified by reducing to rank one or two cases, via a smooth blowing down. In particular, it is proved there that the topological fundamental group $\pi_1(T - SingT)$ is an abelian group of order ≤ 9 .

(6) For an arbitrary log del Pezzo surface T , it is proved in [GZ 1, 2] that the topological fundamental group $\pi_1(T - SingT)$ is finite. But this group may not be abelian, in general; see [Z1] for examples and also the classification, when Y has Picard number one and has at worst one rational triple and several Du Val singular points (97 types altogether).

Lemma 1.11. *Let (X, σ) be as in Lemma 1.1 with $X^\sigma \neq \emptyset$. Suppose that there is an elliptic fibration $\varphi : X \rightarrow \mathbf{P}^1$ such that X^σ is contained in fibers of φ and that there is a σ -stable fiber E_1 with $E_1 \cap X^\sigma \neq \emptyset$. Then S is a smooth rational surface and the following three assertions are true.*

(1) σ induces a permutation among fibers of φ . There are exactly two σ -stable fibers E_1, E_2 of φ . We have $X^\sigma \subseteq \text{Supp}(E_1 + E_2)$.

(2) There is an elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ with π_*E of a general fiber E of φ as a fiber of ψ . The pull back $\pi^*\pi_*E$ is a disjoint union of two smooth fibers E and $\sigma(E)$ of φ .

(3) If mF is a fiber of ψ of multiplicity $m \geq 2$, then $mF = \pi_*E_2$. If F is not a minimal fiber of ψ , then $F = \pi_*E_i$ for $i = 1$ or 2 .

Proof. For every σ -stable fiber E_i (e.g. when $i = 1$), Lemma 1.5 implies that if E contains (resp. does not contain) a component of X^σ then E fits one of Cases

(A), (B), (C) and (D) (resp. is an elliptic curve or fits Case(E)) there and π_*E_i is not a multiple fiber (resp. π_*E_i has multiplicity two). In particular, (3) is a consequence of (1) and Lemma 1.5, together with the multiple-fiber freeness of an elliptic fibration on a K3 surface.

Let E be a general fiber of φ . Then $\sigma(E) \sim \sigma(E_1) = E_1$. So φ induces a permutation among fibers of φ , and also an automorphism on the base curve \mathbf{P}^1 of φ .

Claim(1). σ is not the identity automorphism of \mathbf{P}^1 .

Supppse the contrary that Claim(1) is false. Then every fiber of φ is σ -stable. Moreover, for a general fiber E of φ , the map $\pi : E \rightarrow \pi(E)$ is an etale covering of degree two. For two general fibers E, E' of φ , one has $2\pi(E) = \pi_*E \sim \pi_*E' = 2\pi(E')$. Hence $\pi(E) \sim \pi(E')$ because S is a smooth rational surface (Lemma 1.2). Thus there is an elliptic fibration $h : S \rightarrow \mathbf{P}^1$ with $\pi(E)$ as a general fiber.

Now, $2\pi(E) = \pi_*E \sim \pi_*E_1$. Since the g.c.d. of coefficients in π_*E_1 is 1 (Lemma 1.5), one has $m\pi_*E_1 \sim \pi(E)$ where m (≥ 1) is the multiplicity. We reach a contradiction. This proves Claim(1).

By Claim(1), σ is an automorphism of order 2 on the base curve \mathbf{P}^1 of φ . So, σ has exactly two σ -fixed points by the Hurwitz genus formula for the covering $\mathbf{P}^1 \rightarrow \mathbf{P}^1/\sigma$. Thus, there are exactly two σ -stable fibers E_1, E_2 of φ . This proves (1).

Now we prove (2). Let E be a general fiber of φ . By (1), the map $\pi : E \rightarrow \pi(E) = \pi_*E$ is an isomorphism. For two general fibers E, E' we have $\pi_*E \sim \pi_*E'$. So there is an elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ with π_*E as a fiber and satisfying the conditions in (2). This proves Lemma 1.11.

In the subsequent sections, we shall also prove Theorem 3' below which is stronger than Theorem 3 in the Introduction.

Theorem 3'. *Let (X, σ) be the pair satisfying the hypothesis of Theorem 1 in the Introduction. Let X^σ be the fixed locus and $\rho(X)$ the Picard number. Then the following four assertions are true.*

(1) *X^σ is a disjoint union of m smooth curves for some $0 \leq m \leq 10$ and m can attain any value in this range. If $m = 10$, then (X, σ) has one of the following three types.*

Type(Rat). *X^σ is a union of 10 smooth rational curves. One has $\rho(X) = 20$.*

Type(Gn2). *X^σ is a union of one genus-two curve and 9 smooth rational curves. One has $\rho(X) \geq 18$.*

Type(Ell). *X^σ is a union of one elliptic curve E and 9 smooth rational curves. One has $\rho(X) \geq 19$.*

(2) *There is, up to isomorphisms, only one pair (X, σ) of Type(Rat). Such a pair is called Shioda-Inose's pair in [OZ1, Example 2] (see [Z2, Example 3.2] and Example 2.8 below for two more constructions, of the same pair). In particular, X is isomorphic to the unique K3 surface with the discriminant of $\text{Pic} X$ equal to 4*

(see [V] for $\text{Aut} X$).

(3) There is a family $\mathcal{X}_{\text{ell}} := \{(X_s, \sigma_s) | s \in \mathbf{P}^1\}$ of surface X_s and an involution σ_s on it, satisfying the following four assertions (see the proof for the construction).

(3i) X_s/σ_s is equal to S_{ell} for all s with a smooth rational surface S_{ell} , which is independent of s and given in the proof. Let $\pi : X_s \rightarrow S_{\text{ell}}$ be the quotient morphism.

(3ii) For $s = \infty$, $X_\infty = S_{\text{ell}} \amalg S_{\text{ell}}$ is a disjoint union and σ_∞ is the order switching.

(3iii) For three fixed distinct points $s = 0, 1, s_0$, the pair (Y_s, σ_s) is obtained from Shioda-Inose's pair (X, σ) so that $X \rightarrow X_s$ is the contraction of a $(\sigma\text{-stable})$ smooth rational curve on X meeting at two distinct points with a σ -fixed curve which is hence mapped to a 1-nodal curve E_s on X_s (the node = $\text{Sing } X_s$), and that σ_s is induced from σ . The fixed locus $X_s^{\sigma_s}$ is a disjoint union of E_s and 9 smooth rational curves.

(3iv) For $s \neq \infty, 0, 1, s_0$, the pair (X_s, σ_s) is of $\text{Type}(\text{Ell})$. Conversely, every pair of $\text{Type}(\text{Ell})$ is isomorphic to (X_s, σ_s) for some $s \neq \infty, 0, 1, s_0$.

(4) There is a family $\mathcal{Y}_{\text{gn2}} = \{(Y_t, \sigma_t) | t \in \mathbf{P}^3\}$ of surface Y_t and an involution σ_t on it, satisfying the following three assertions (see the proof for the construction).

(4i) Y_t/σ_t is equal to S_{gn2} for all t with a smooth rational surface S_{gn2} , which is independent of t and given in the proof.

(4ii) For a general t , the pair (Y_t, σ_t) is of $\text{Type}(\text{Gn2})$. Conversely, every pair of $\text{Type}(\text{Gn2})$ is isomorphic to (Y_t, σ_t) for some t .

(4iii) There is a straight projective line B in \mathbf{P}^3 such that the subset $\overline{\mathcal{X}}_{\text{ell}} := \{(\overline{X}_s, \sigma_s) := (Y_s, \sigma_s) | s \in B = \mathbf{P}^1\}$ is obtained from the family \mathcal{X}_{ell} , in the following way:

For $s = \infty$, $\overline{X}_s = S_{\text{gn2}} \amalg S_{\text{gn2}}$ is a disjoint union and σ_∞ is the order switching. $S_{\text{ell}} \rightarrow S_{\text{gn2}}$ is the smooth blowing-down of a (-1) -curve H_0 .

For $s \neq \infty$, $X_s \rightarrow \overline{X}_s$ is the contraction of the σ_s -stable divisor $\pi^* H_0$ on X_s meeting (with intersection two) with $X_s^{\sigma_s}$'s only arithmetic-genus 1 curve E_s which is hence mapped to an arithmetic genus 2 curve \overline{E}_s with $\text{Sing } \overline{E}_s = \text{Sing } \overline{X}_s$, and that the σ_s on \overline{X}_s is induced from the σ_s on X_s . The set $\overline{X}_s^{\sigma_s}$ is a disjoint union of \overline{E}_s and 9 smooth rational curves.

To be precise, for some $s_1 \in \mathbf{P}^1 - \{\infty, 0, 1, s_0\}$, $\pi^* H_0$ is a linear chain of two (-2) -curves and E_{s_1} is a smooth elliptic curve through (transversally) the intersection of $\pi^* H_0$, and for all $s \in \mathbf{P}^1 - \{\infty, s_1\}$, $\pi^* H_0$ is a (-2) -curve intersecting E_s at its two distinct smooth points.

§2. Rational log Enriques surfaces of index 2

Let R be a rational log Enriques surface of index 2. We shall use the notation in Lemma 1.4 :

$$q_1(= g_1) : S_1 \rightarrow R(= T), \quad q_2(= g_2) : S \rightarrow S_1, \quad q = q_1 \circ q_2, \quad K_S = q^* K_R - 1/2 \sum_{i=1}^r \sum_{j=1}^{n_i} D_{i,j}.$$

Now the following relation

$$(2, 1, 1) \cdot \mathcal{O}(K_S)^{\otimes 2} \sim \mathcal{O}$$

induces a relation:

$$(2.1.2) \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}\left(\sum_{i=1}^r \sum_{j=1}^{n_i} D_{i,j}\right).$$

Let $\bar{\pi} : W := \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_R) \rightarrow R$ and $\pi : X := \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_S) \rightarrow S$ be the double coverings associated with the relations (2.1.1) and (2.1.2), respectively. Then both $\text{Gal}(W/R)$ and $\text{Gal}(X/S)$ are isomorphic to the same cyclic group $\langle \sigma \rangle$ of order 2 such that $W/\sigma = R$ and $X/\sigma = S$.

Lemma 2.1. *Let R be a rational log Enriques surface of index 2. Let $\bar{\pi} : W \rightarrow R = W/\sigma$ and $\pi : X \rightarrow S = X/\sigma$ be as above. Then the following five assertions hold true.*

(1) *One has $\pi^* D_{i,j} = 2C_{i,j}$ for a smooth rational curve $C_{i,j}$. The fixed locus X^σ is a disjoint union of n smooth rational curves $C_{i,j}$, where $n := \sum_{i=1}^r n_i$.*

(2) *X is a smooth K3 surface. The involution σ on X satisfies $\sigma^* \omega = -\omega$ for a non-zero holomorphic 2-form ω on X .*

(3) *There exists a σ -equivariant birational morphism $p : X \rightarrow W$ which induces the following commutative diagram with p as the minimal resolution and q a resolution*

$$\begin{array}{ccc} X & \xrightarrow{p} & W \\ \pi \downarrow & & \downarrow \bar{\pi} \\ S & \xrightarrow{q} & R. \end{array}$$

(4) *There are $n - 1$ smooth rational curves G_j such that $\Gamma := \sum_{i,j} C_{i,j} + \sum_j G_j$ is a component-wisely σ -stable linear chain of length $2n - 1$ as follows:*

$$C_1 - G_1 - C_2 - G_2 - \cdots - C_{n-1} - G_{n-1} - C_n.$$

Here $\{C_{i,j}\} = \{C_k\}$ as sets. In particular, $\text{rank Pic} X \geq 2n$ and $1 \leq r \leq n \leq 10$.

Let $u_2 : X \rightarrow X_2$ be the contraction of Γ into a type A_{2n-1} Du Val singular point. Then $S_2 := X_2/\sigma$ is a rational log Enriques surface of index 2 and Type A_{2n-1} (Def. 1.7).

(5) *Suppose in addition that $n = 10$. Then S_2 is isomorphic to the unique rational log Enriques surface of Type A_{19} (see [OZ1, Theorem 2 and Example 2], [Z2, Example 3.2] and Example 2.8 below for the proof of the uniqueness and three different constructions, of the same surface). In particular, X is isomorphic to the unique K3 surface with the discriminant of $\text{Pic} X$ equal to 4.*

Proof. (1) Note that X is a smooth surface and π is ramified exactly over the disjoint union $\coprod D_{i,j}$. So $\pi^* D_{i,j} = 2C_{i,j}$ for a smooth rational curve $C_{i,j}$. Now (1) follows.

(2) By the ramification formula, one has $K_X \sim 0$. Thus X is a smooth K3 surface because X is a double covering of a rational surface (cf. [TY, Theorem 0.1]).

Since $\sigma^2 = \text{id}$, one has $\sigma^* \omega = \pm \omega$. If $\sigma^* \omega = \omega$, then σ acts trivially on $H^2(X, \mathcal{O}_X) = \mathbf{C}\bar{\omega}$. Hence $H^2(S, \mathcal{O}_S) = H^2(X, \mathcal{O}_X)^\sigma \cong \mathbf{C}$. This contradicts the rationality of S because $h^2(S, \mathcal{O}_S) = h^0(S, K_S) = 0$.

(3) follows from the constructions of $\bar{\pi}$, π .

(4) Let $v_3 : S \rightarrow S_3$ be the contraction of n (-4) -curves $D_{i,j}$ into n cyclic quotient singular points of Brieskorn type $C_{4,1}$. By the relation (2.1.2), one has $\mathcal{O}(-K_{S_3})^{\otimes 2} \cong \mathcal{O}_{S_3}$. Hence, S_3 is a rational log Enriques surface of index 2. Now applying [Z2, Theorem 3.6] to $\bar{V} = S_3$ and $(V, D) := (S, \sum_{i,j} D_{i,j})$, one sees that there are $n - 1$ (-1) -curves H_j such that $\Delta := \sum_{i,j} D_{i,j} + \sum_j H_j$ is a linear chain as follows, where $\{D_{i,j}\} = \{D_k\}$ as sets

$$D_1 - H_1 - D_2 - H_2 - \cdots - D_{n-1} - H_{n-1} - D_n.$$

The pull back by π^* of this linear chain gives the linear chain Γ in (4), where $G_j := \pi^* H_j$. Now the first paragraph of (4) is clear, while the second paragraph follows from the observation that X_2 is a single singular point $u_2(\Gamma)$.

(5) follows from (4) and [OZ1, Theorem 2 and Example 2].

Definition 2.2. In Lemma 2.1, the double covering $\pi : X \rightarrow S$ is called *the canonical resolution* of the double covering $\bar{\pi} : W \rightarrow R$.

The following Lemma 2.3 is the converse to Lemma 2.1.

Lemma 2.3 *Let (X, σ) be as in Theorem 1. Assume further that X^σ is a disjoint union of n ($n \geq 1$) smooth rational curves.*

Then the pair (X, σ) can be realized in the way of Lemma 2.1, from a rational log Enriques surface R of index 2 and Type A_{2n-1} with a cyclic quotient singularity of Brieskorn type $C_{4n, 2n-1}$ as its only singular point, i.e., the quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution of the double covering $\bar{\pi} : W \rightarrow R$ there.

Proof. By Lemma 1.2 (2), the quotient morphism $\pi : X \rightarrow S = X/\sigma$ coincides with the double covering associated with the relation (2.1.2) where $\sum_{i,j} D_{i,j} = \pi(X^\sigma)$. By the proof of Lemma 2.1 (4), there are $n - 1$ (-1) -curves H_j on S such that $\Delta := \sum_{i=1}^n D_i + \sum_{j=1}^{n-1} H_j$ is a linear chain as shown there, where $\sum_{i,j} D_{i,j} = \sum_{k=1}^n D_i$.

Let $q : S \rightarrow R$ be the contraction of Δ into a cyclic-quotient singular point of Brieskorn type $C_{4n, 2n-1}$. Then our relation (2.1.2) induces the relation (2.1.1). Hence R is a rational log Enriques surface of index 2 (Lemma 1.2 (4)).

Clearly, π is the canonical resolution of the canonical double covering $\bar{\pi} : W \rightarrow R$ associated with the relation (2.1.1), because our contraction $q : S \rightarrow R$ coincides with the map $q(=g) : S \rightarrow R(=T)$ constructed in Lemma 1.4. This proves Lemma 2.3.

We now construct the pair (X, σ) of Lemma 2.3 in a way different from Lemma 2.1.

Let S_c be a smooth rational surface with a relatively minimal elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$. Suppose that ψ has two fibers $\bar{F}_1, m_2 \bar{F}_2$ such that either one of the following two cases occurs

Case(a). $m_2 = 1$, ψ is multiple-fiber free, and each \overline{F}_i is either one of Kodaira types II, III, IV and I_{n_i} ($n_i \geq 1$).

Case(b). $m_2 = 2$, \overline{F}_2 is of Kodaira type I_{s_2} ($s_2 \geq 0$), the multiplicity-two fiber $2\overline{F}_2$ is the only multiple fiber of ψ , and \overline{F}_1 is either one of Kodaira types II, III, IV and I_{n_1} .

In Case(a) (resp. Case(b)), let $g : S \rightarrow S_c$ be the blowing-up of intersection points in \overline{F}_i for $i = 1$ and 2 (resp. for $i = 1$ only) and their infinitely near points so that $F_i := g^*\overline{F}_i$ fits one of Cases (α) , (β) , (γ) and (δ) in Lemma 1.5, according to the type of \overline{F}_i . We shall use the notation $\text{Supp}F_i = \sum_{j=1}^{n_i} D_{i,j} + \sum_j H_{i,j}$ there. In Case(b) we let $F_2 := g^*\overline{F}_2$, where F_2 ($\cong \overline{F}_2$) is of Kodaira type I_{s_2} .

In Case(a) and Case(b), the canonical divisor formula implies respectively the following two relations:

$$(2.4.1a) \quad \mathcal{O}(-K_{S_c})^{\otimes 2} \cong \mathcal{O}(\overline{F}_1 + \overline{F}_2),$$

$$(2.4.1b) \quad \mathcal{O}(-K_{S_c})^{\otimes 2} \cong \mathcal{O}(\overline{F}_1).$$

Now (2.4.1a) and (2.4.1b) induce respectively the following two relations:

$$(2.4.2a) \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}\left(\sum_{i=1}^2 \sum_{j=1}^{n_i} D_{i,j}\right),$$

$$(2.4.2b) \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}\left(\sum_{j=1}^{n_1} D_{1,j}\right).$$

In Case(a) (resp. Case(b)), let $\pi_c : X_c = \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_{S_c}) \rightarrow S_c$ be the double covering associated with the relation (2.4.1a) (resp. (2.4.1b)), and let $\pi : X = \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_S) \rightarrow S$ be the double covering associated with the relation (2.4.2a) (resp. (2.4.2b)). Then both $\text{Gal}(X_c/S_c)$ and $\text{Gal}(X/S)$ are isomorphic to the same cyclic group $\langle \sigma \rangle$ of order 2 such that $X_c/\sigma = S_c$ and $X/\sigma = S$.

Lemma 2.4. *Let S_c be a smooth rational surface with a relatively minimal elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$ fitting the above Case(a) (resp. Case(b)). Let $\pi_c : X_c \rightarrow S_c = X_c/\sigma$ and $\pi : X \rightarrow S = X/\sigma$ be as above. Then the following four assertions hold true.*

(1) *For both $i = 1$ and 2 (resp. $i = 1$ only), one has $\pi^*D_{i,j} = 2C_{i,j}$ for a smooth rational curve $C_{i,j}$ ($j = 1, 2, \dots, n_i$), and the fixed locus X^σ is a disjoint union of $n := n_1 + n_2$ (resp. $n := n_1$) curves $C_{i,j}$ contained in “the fiber” E_i (see (4) below). One has the Picard number $\rho(X) \geq \rho(S) = 10 + n$, and $2 \leq n \leq 10$ (resp. $1 \leq n \leq 9$).*

(2) *X is a smooth K3 surface. The involution σ on X satisfies $\sigma^*\omega = -\omega$ for a non-zero holomorphic 2-form ω on X .*

(3) *There exists a σ -equivariant birational morphism $f : X \rightarrow X_c$ which induces the following commutative diagram with f as the minimal resolution and g a resolution*

$$\begin{array}{ccc} & & \\ & f & \\ \mathbf{Y} & & \mathbf{Y}_c \end{array}$$

$$\begin{array}{ccc} \pi & \downarrow & \downarrow \pi_c \\ & S & \xrightarrow{g} S_c. \end{array}$$

(4) For both $i = 1$ and 2 (resp. $i = 1$ only), one has $\pi^*F_i = 2E_i$ where E_i fits one of Cases (A), (B), (C) and (D) in Lemma 1.5 according to the type of F_i . In Case(b), one has $\pi^*F_2 = E_2$ where E_2 is of Kodaira type I_{2s_2} .

Proof. The first three assertions, except for the last part of (1), can be proved similarly as in Lemma 2.1, while the assertion(4) follows from the construction of π .

Note that $\rho(S) = 10 - K_S^2 = 10 + n$ by (2.4.2).

When \overline{F}_i is of Kodaira type II, III or IV one has $n_i = 1, 2$ or 4 respectively. Now assume that \overline{F}_i has Kodaira type I_{n_i} . Since $\psi : S_c \rightarrow \mathbf{P}^1$ is relatively minimal, one has $K_{S_c}^2 = 0$ and the Picard number $\rho(S_c) = 10$. Now the inequalities for n follow from the inequality $\rho(S_c) \geq 2 + \sum_i (\#(\text{irreducible components in } \overline{F}_i) - 1)$. This proves Lemma 2.4.

In the sense of Horikawa [H1, p.48], we make the following:

Definition 2.5. In Lemma 2.4, the double covering $\pi : X \rightarrow S$ is called *the canonical resolution* of the covering $\pi_c : X_c \rightarrow S_c$.

Remark 2.6. The pair (n_1, n_2) in Lemma 2.4 Case(a), satisfies:

$$n_i \geq 1, \quad 2 \leq n_1 + n_2 \leq 10.$$

On the other hand, [P, the list] (see also [Mi]) classified the set of singular fibers of a rational relatively minimal elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$ with no multiple fibers. Going through Persson's list, we see that all pairs (n_1, n_2) satisfying the above conditions, can be obtained in the way of Lemma 2.4. In particular, $n := n_1 + n_2$ in Lemma 2.4 Case(a), can attain any value in the range $2 \leq n \leq 10$.

Example 2.7. Let $\overline{D} \subseteq \mathbf{P}^1$ be a rational curve of degree 6 with 10 ordinary nodes as its only singular points. Let $h : S \rightarrow \mathbf{P}^1$ be the blowing-up of all these 10 nodes and let $D = h^*\overline{D}$ be the proper transform. Then one has $\mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}(D)$.

Denote by $q : S \rightarrow R$ the contraction of D into a cyclic quotient singular point of Brieskorn type $C_{4,1}$. Then one has the relation $\mathcal{O}(-K_R)^{\otimes 2} \cong \mathcal{O}_R$. a rational log Enriques surface of index 2 with exactly one singular point.

Example 2.8. By [P, the list], there is a smooth rational surface S_c with a multiple-fiber free relatively minimal elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$ such that ψ has two singular fibers \overline{F}_i ($i = 1, 2$) of Kodaira type I_{n_i} , where (n_1, n_2) can be taken as any of the following three pairs: $(1, 9), (2, 8), (5, 5)$ (cf. Lemma 7 in §7). In particular, we have $n_1 + n_2 = 10$.

We use the notation in Lemma 2.4 Case(a): $g : S \rightarrow S_c$, $F_i = g^*\overline{F}_i = D_{i,1} + 2H_{i,1} + D_{i,2} + 2H_{i,2} + \cdots + D_{i,n_i-1} + 2H_{i,n_i-1} + D_{i,n_i}$, and the canonical resolution $\pi : X \rightarrow S$ of the double covering $\pi_c : X_c \rightarrow S_c$.

Let M' be a (-1) -curve on S_c . The relation (2.4.1a) implies that $M' \cdot \overline{F}_i = 1$. Let $M = g^{-1}M'$, which is a (-1) -curve on S . We may assume that $M \cdot D_{1,n_1} = M \cdot D_{2,1} = 1$, after relabelling. Thus we obtain a linear chain of 10 (-4) -curves $D_{i,j}$ and 9 (-1) -curves M and $H_{i,j}$ as follows:

$$D_{1,1} - H_{1,1} - D_{1,2} - H_{1,2} - \cdots - D_{1,n_1} - M - D_{2,1} - H_{2,1} - D_{2,2} - H_{2,2} - \cdots - D_{2,n_2}.$$

Actually, in view of Lemmas 2.1(4), 2.3 and 2.4, starting with any two fibers \overline{F}_i of Kodaira type II (setting $n_i := 1$), III ($n_i := 2$), IV ($n_i := 4$) or I_{n_i} with $n_1 + n_2 = 10$, we can obtain a linear chain of 10 (-4) -curves and 9 (-1) -curves as shown above, after a suitable relabelling.

(2.8.1). Let $v_2 : S \rightarrow S_2$ be the contraction of this linear chain of length 19 into a cyclic quotient singular point of Brieskorn type $C_{40,19}$ (Lemma 1.4). Then S_2 is isomorphic to the *unique* rational log Enriques surface of Type A_{19} (see [OZ1, Theorem 2 and Example 2] and [Z2, Example 3.2] for the proof of the uniqueness and two different constructions, of the same surface). By Definition 2.2, our π is also the canonical resolution of the canonical covering $\overline{\pi}_2 : X_2 \rightarrow S_2$.

(2.8.2). Let $v_3 : S \rightarrow S_3$ be the contraction of 10 disjoint (-4) -curves $D_{i,j}$ into 10 cyclic-quotient singular points of Brieskorn type $C_{4,1}$. Then S_3 is the *unique* rational log Enriques surface of index 2 with 10 Cartier-index two singular points (Corollary 5). We note that $\sum_{i,j} D_{i,j} = \pi(X^\sigma)$ (Lemma 2.4).

(2.8.3). Since X^σ is a disjoint union of 10 smooth rational curves $C_{i,j} := \pi^{-1}(D_{i,j})$, the pair (X, σ) here is isomorphic to Shioda-Inose's unique pair in Theorem 3. In particular, $S = X/\sigma$ is independent of the choice of fibers \overline{F}_i so long as $n_1 + n_2 = 10$.

Lemma 2.9. *Let (X, σ) be as in Theorem 1. Assume further that X^σ is a disjoint union of n ($n \geq 1$) smooth rational curves C_i .*

Then the pair (X, σ) can be realized in the way of Lemma 2.4, from a rational surface S_c satisfying all hypotheses there, i.e., our quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution of the double covering $\pi_c : X_c \rightarrow S_c$ there. Moreover, each fiber \overline{F}_i there can be so chosen that it is not of Kodaira type IV.

Proof. By Lemma 2.3, the quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution, in the sense of Definition 2.2 or Lemma 2.1, of the canonical covering $\overline{\pi} : W \rightarrow R$ of a rational log Enriques surface R of index 2 with a cyclic-quotient singular point of Brieskorn type $C_{4n,2n-1}$ as its only singular point.

We use the notation in Lemma 2.1 : $q_1 : S_1 \rightarrow R$, $q_2 : S \rightarrow S_1$, $q = q_1 \circ q_2$, and the commutative diagram there. Note that $\Delta := q^{-1}(\text{Sing} R) = \sum_{i=1}^n D_i + \sum_{j=1}^{n-1} H_j$ is a linear chain as shown in the proof of Lemma 2.1(4); X_2 , S_2 there and W , R there coincide in the present case. q_2 is just the smooth blowing-down of $n-1$ (-1) -curves H_j in Δ .

By [GZ3] or [Z6, Proposition 3.1], there exists a (-1) -curve M' on S_1 such that M' meets a tip component of the linear chain Δ . Note that $M' \cdot \Delta = 2$

because of the relation $-2K_{S_1} \sim q_2(\Delta)$ deduced from (2.1.2). Let $M := q'_2 M'$ be the proper transform on S . Then one of the following three cases occurs.

Case(α)'. $M^2 = -1$, $M \cdot \Delta = 2$ and M has an order-two touch with D_1 at a point $d_1 \in D_1 - D_2$. In this case we set $n_1 := 1$.

Case(β)'. $M^2 = -2$, $M \cdot \Delta = 1$, and M meets H_1 at a point $d_1 \in H_1 - (D_1 + D_2)$. In this case we set $n_1 := 2$.

Case(δ)'. $M^2 = -1$, $M \cdot \Delta = 2$ and M meets two distinct points d_1, d_2 of $\sum_i D_i$. To be precise, $d_1 \in D_1$, $d_2 \in D_{n_1}$ ($n_1 \geq 1$).

In Case (α)', (β)' and (δ)', let $F_1 := 2M + D_1$, $F_1 := 4H_1 + 2M + D_1 + D_2$, $F_1 := D_1 + 2H_1 + D_2 + 2H_2 + \cdots + D_{n_1-1} + 2H_{n_1-1} + D_{n_1} + 2M$, respectively. Then F_1 fits respectively Case (α), or (β), or (δ) of Lemma 1.5, so that F_1 contains $\sum_{i=1}^{n_1} D_i$.

Claim(1). There is an elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ with F_1 as a non-multiple fiber.

By the construction of π , one has $\pi^* F_1 = 2E_1$ where E_1 fits Case (A), (B) or (D) in Lemma 1.5 according to the type of F_1 . By the Riemann-Roch theorem, there exists an elliptic fibration $\varphi : X \rightarrow \mathbf{P}^1$ with E_1 as a fiber. Clearly, $X^\sigma = \sum_{i=1}^{n_1} C_i$, where $\pi^* D_i = 2C_i$, is contained in fibers of φ and E_1 is σ -stable. So one can apply Lemma 1.11. Thus Claim(1) follows.

Claim(2). Let E_1, E_2 be the only σ -stable fibers of φ (Lemma 1.11). Then E_2 does not fit Case(C) in Lemma 1.5.

If $n_1 = n$ then E_2 contains no component of X^σ and hence E_2 is of Kodaira type I_{s_2} . If $n > n_1$, all components of $E_2 \cap X^\sigma$ are contained in the linear chain $\pi^{-1}(D_{n_1+1} + H_{n_1+1} + \cdots + D_n)$ which is a subset of E_2 . Then Claim(2) is clear because there is no such a linear chain in E_2 fitting Case(C).

By Lemma 1.5 and Claim(2), E_2 fits Case (a) or (b) below. Let $g : S \rightarrow S_c$ be the smooth blowing-down of curves in fibers $\pi_* E_i$ so that $\psi : S \rightarrow \mathbf{P}^1$ induces a relatively minimal elliptic fibration, also denoted by $\psi : S_c \rightarrow \mathbf{P}^1$ (Lemma 1.11 (3)). Then $\overline{F}_1 := g_* F_1$ has respectively Kodaira type (II), (III) or I_{n_1} .

Case(a). E_2 fits one of Cases (A), (B) and (D) in Lemma 1.5. Then $F_2 := \pi_* E_2$ is a non-multiple fiber of ψ fitting respectively one of Cases (α), (β) and (δ) in Lemma 1.5. Hence ψ is multiple fiber free (Lemma 1.11 (3)). Moreover, $\overline{F}_2 := g_* F_2$ has respectively one of Kodaira types II, III and I_{n_2} ($n_2 \geq 1$), where $n = n_1 + n_2$.

Case(b). E_2 is of Kodaira type I_{2s_2} ($s_2 \geq 0$). To be precise, E_2 is either an elliptic curve or fits Case(E) in Lemma 1.5. So $\pi_* E_2 = 2F_2$ where F_2 is of Kodaira type I_{s_2} . Hence $2E_2$ or $2\overline{F}_2$ where $\overline{F}_2 := g_* F_2$ ($\approx F_2$) is the only multiple fiber of

ψ (Lemma 1.11 (3)). One has $n = n_1$.

Thus S_c satisfies all hypotheses in Lemma 2.4. By Lemma 1.2 (2), our quotient morphism $\pi : X \rightarrow S = X/\sigma$ is the canonical resolution of the double covering $\bar{\pi}_c : X_c \rightarrow S_c$ associated with the relation (2.4.1a) or (2.4.1b), respectively, because our blowing-down $g : S \rightarrow S_c$ coincides with the map $g : S \rightarrow S_c$ constructed preceding Lemma 2.4. This proves Lemma 2.9.

Definition and Proposition 2.10. In Lemma 2.9, the double covering $\pi_c : X_c \rightarrow S_c$ is called a *canonical mapping model* of $\pi : X \rightarrow S$. This π_c is not unique (see Example 2.8), for there is no unique elliptic fibration $\varphi : X \rightarrow \mathbf{P}^1$ such that X^σ is contained in fibers of φ . One may have also noticed that the fiber \bar{F}_i in Lemma 2.4 can be taken as of Kodaira type IV, but we can avoid this type in Lemma 2.9.

§3. Log del Pezzo surfaces of Cartier index ≤ 2

Let S_c be a log del Pezzo surface of Cartier-index ≤ 2 . We shall use the notation in Lemma 1.4, where $r \geq 0$:

$$g_1 : S_1 \rightarrow S_c (= T), \quad g_2 : S \rightarrow S_1, \quad g = g_1 \circ g_2, \quad K_S = g^* K_{S_c} - 1/2 \sum_{i=1}^r \sum_{j=1}^{n_i} D_{i,j}.$$

Lemma 3.1. *Let S_c be a log del Pezzo surface of Cartier-index ≤ 2 . Then the following two assertions are true.*

- (1) $|-2K_{S_c}|$ contains a smooth curve \bar{F} of genus ≥ 2 with $\bar{F} \cap \text{Sing} S_c = \emptyset$.
- (2) $\dim |-2K_{S_c}| = 3K_{S_c}^2 = 3(n + K_S^2)$, where $n := \sum_{i=1}^r n_i$.

Proof. (1) When S_c has Cartier-index 2, (1) is proved in [AN, Theorem 3].

Now assume that S_c is a log del Pezzo surface of Cartier-index one, i.e., S_c has at worst Du Val singular points. Then $-K_S = -g^* K_{S_c}$ is nef and big, and has zero intersection with $g^{-1}(\text{Sing} S_c)$. By [D, Theorem 1, p. 55], $|-2K_S|$ is base point free. Actually, [D] assumed the condition that S is a blowing-up of several points on \mathbf{P}^2 . However, if this condition is not true then S is the Hirzebruch surface of degree 2 and $g : S \rightarrow S_c$ is the contraction of the (-2) -curve. In this case, $|-2K_{S_c}|$ is also base point free.

So a general member F of $|-2K_S|$ is smooth and also connected because $(-2K_S)^2 > 0$. Now $\bar{F} := g_* F$ satisfies the conditions in (1). Indeed, $g(\bar{F}) = g(F) \geq 2$ comes from the genus formula. This proves (1).

(2) Set $F := g^* \bar{F} \sim g^*(-2K_{S_c})$. Since S_c has only rational singularities and by the projection formula, one has $R^i g_* \mathcal{O}(F) \cong \mathcal{O}(\bar{F}) \otimes R^i g_* \mathcal{O}_S = 0$ for all $i \geq 0$. Hence $H^i(S, \mathcal{O}(F)) \cong H^i(S_c, g_* \mathcal{O}(F)) = H^i(S_c, \mathcal{O}(\bar{F}))$ for all i . Since $\bar{F} - K_{S_c} \sim -3K_{S_c}$ is \mathbf{Q} -ample, [KMM, Theorem 1-2-5] implies that $H^i(S_c, \bar{F}) = 0$ for all $i \geq 1$.

Now the Riemann-Roch theorem implies that $h^0(S_c, -2K_{S_c}) = h^0(S, F) = 1 + 1/2 c_2^*(-2K_{S_c}) (c_1^*(-2K_{S_c}) - K_{S_c}) = 1 + 1/2 c_2^*(-2K_{S_c}) (c_1^*(-2K_{S_c}) - c_1^* K_{S_c})$.

$1+3K_{S_c}^2$. By the ramification formula preceding Lemma 3.1, one has $K_S^2 = K_{S_c}^2 - n$. Now (2) follows. This completes the proof of Lemma 3.1.

Let \overline{F} be as in Lemma 3.1. Then one has:

$$(3.2.1) \quad \mathcal{O}(-K_{S_c})^{\otimes 2} \cong \mathcal{O}(\overline{F}).$$

Let $F := g^*\overline{F}$, which is a smooth curve away from $g^{-1}(\text{Sing}S_c)$. Now (3.2.1) induces:

$$(3.2.2) \quad -2K_S \sim g^*(-2K_{S_c}) + \sum_{i=1}^r \sum_{j=1}^{n_i} D_{i,j} \sim F + \sum_{i,j} D_{i,j}, \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}(F + \sum_{i,j} D_{i,j}).$$

Here $F + \sum_{i,j} D_{i,j}$ is a disjoint union of F and n (-4) -curves $D_{i,j}$, where $n = \sum_{i=1}^r n_i$. Let $\pi_c : X_c := \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_{S_c}) \rightarrow S_c$ and $\pi : X := \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_S) \rightarrow S$ be the double coverings associated with the relations (3.2.1) and (3.2.2), respectively. Then both $\text{Gal}(X_c/S_c)$ and $\text{Gal}(X/S)$ are isomorphic to the same cyclic group $\langle \sigma \rangle$ of order 2 such that $X_c/\sigma = S_c$ and $X/\sigma = S$.

Lemma 3.2. *Let S_c be a log del Pezzo surface of Cartier-index ≤ 2 . Let $\pi_c : X_c \rightarrow S_c = X_c/\sigma$ and $\pi : X \rightarrow S$ be as above. Then the following three assertions hold true.*

(1) *One has $\pi^*F = 2E$ for a smooth curve E isomorphic to F , and $\pi^*D_{i,j} = 2C_{i,j}$ for a smooth rational curve $C_{i,j}$. The fixed locus X^σ is a disjoint union of E and n curves $C_{i,j}$, where $n := \sum_{i=1}^r n_i$. One has $n \leq 9$.*

(2) *X is a smooth K3 surface. The involution σ on X satisfies $\sigma^*\omega = -\omega$ for a non-zero holomorphic 2-form ω on X .*

(3) *There exists a σ -equivariant birational morphism $f : X \rightarrow X_c$ which induces the following commutative diagram with f as the minimal resolution and g a resolution*

$$\begin{array}{ccc} X & \xrightarrow{f} & X_c \\ \pi \downarrow & & \downarrow \pi_c \\ S & \xrightarrow{g} & S_c. \end{array}$$

(4) *Suppose in addition that $n = 9$. Then S_c is isomorphic to the unique log del Pezzo surface of Picard number 1 and Cartier index 2 (see §7 for the construction and [AN, Figure 1] for the configuration of all 19 exceptional curves on S).*

In particular, $\text{Sing}S_c$ is a single cyclic quotient singular point of Brieskorn type $C_{36,17}$, $g(E) = 2$ and the Picard number $\rho(S) = 18$, whence $\rho(X) \geq 18$. Moreover, $\mathcal{O}(X^\sigma) = \mathcal{O}(E + \sum_{i,j} D_{i,j})$ is divisible by 2 in $\text{Pic}S$ and hence S is the quotient of a smooth surface of general type, modulo an involution.

Proof. Except for the inequality $n \leq 9$, assertions (1), (2) and (3) can be proved similarly as in Lemma 2.1. If S_c has Cartier index 1 then $n = 0$.

If S_c has Cartier index 2, then $1 \leq n \leq 9$ by [AN, Theorem 4]. Suppose that $n = 0$. Then (4) follows from [AN, Theorem 4, (7)] because there is apparently no

proper DPN-subgraph $\Gamma(S)$ of $\Gamma(\tilde{S})$, where \tilde{S} is extremal with $(n, g(E), \delta) = (9, 2, 0)$ in the notation there, due to the fact that $DuVal\Gamma(\tilde{S}) = \phi$.

Definition 3.3. In Lemma 3.2, the double covering $\pi : X \rightarrow S$ is called *the canonical resolution* of the covering $\pi_c : X_c \rightarrow S_c$.

The following Lemma 3.4 is the converse to Lemma 3.2.

Lemma 3.4. *Let (X, σ) be as in Theorem 1. Assume further that X^σ contains a smooth curve E of genus ≥ 2 .*

Then X^σ is a disjoint union of E and n ($n \geq 0$) smooth rational curves, and the pair (X, σ) can be realized in the way of Lemma 3.2, from a log del Pezzo surface S_c of Cartier index ≤ 2 , i.e., the quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution of the double covering $\pi_c : X_c \rightarrow S_c$ there.

Proof. Let Σ be the set of all curves on X which have zero intersection with E . Since $E^2 = 2g(E) - 2 > 0$, the Hodge index theorem and the finiteness of rank $PicX$ imply that Σ consists of finitely many curves and has negative definite intersection matrix. In particular, every curve in Σ is a smooth rational curve by the genus formula. Thus every connected component of Σ is disjoint from E and has Dynkin type A_m ($m \geq 1$), D_m ($m \geq 4$) or E_m ($m = 6, 7, 8$).

Let Γ_i ($i = 1, 2, \dots, r; r \geq 0$) be all connected components of Σ containing a curve of X^σ . Let Δ_j ($j = 1, 2, \dots, s_1; s_1 \geq 0$) be the remaining connected components of Σ .

Since E is σ -fixed, σ induces a permutation on the set of connected components of Σ . So Γ_i is σ -stable, while $\sigma(\Delta_j) = \Delta_{j'}$ for some $j' \neq j$ (Lemma 1.6). Hence we may assume that $s_1 = 2s$ and $\sigma(\Delta_i) = \Delta_{s+i}$ ($i = 1, 2, \dots, s$).

On the other hand, by Lemma 1.6, Γ_i is of type A_{2n_i-1} ($n_i \geq 1$) as follows:

$$C_{i,1} - G_{i,1} - C_{i,2} - G_{i,2} - \dots - C_{i,n_i-1} - G_{i,n_i-1} - C_{i,n_i}.$$

Here $C_{i,j}$ is σ -fixed, while $G_{i,j}$ is σ -stable but not σ -fixed.

Note that X^σ is the disjoint union of E and n smooth rational curves $C_{i,j}$, where $n = \sum_{i=1}^r n_i$. So, by Lemma 1.2, the quotient morphism $\pi : X \rightarrow S = X/\sigma$ coincides with the double covering associated with the relation (3.2.2) where $F := \pi_*E = \pi(E)$, $D_{i,j} := \pi_*C_{i,j} = \pi(C_{i,j})$.

Since $\pi^*F = 2E$, the following is clear.

Claim(1). $\pi(\Sigma)$ consists of exactly all curves on S having zero intersection with F .

Clearly, $\pi(\Delta_j) = \pi(\Delta_{s+j})$ ($j = 1, 2, \dots, s$) is a connected component of $\pi(\Sigma)$ disjoint from F and with the same weighted dual graph as Δ_j .

On the other hand, $\pi(\Gamma_i)$ ($i = 1, 2, \dots, r$) is a connected component of $\pi(\Sigma)$ disjoint from F and with the following dual graph :

Here $D_{i,j}$ is a (-4) -curve, while $H_{i,j} := \pi(G_{i,j})$ is a (-1) -curve.

Let $g : S \rightarrow S_c$ be the contraction of $\pi(\Sigma)$ into points. Then $t_i := g\pi(\Gamma_i)$ ($i = 1, 2, \dots, r$) is a cyclic quotient singularity of Brieskorn type $C_{4n_i, 2n_i-1}$ (Lemma 1.4), while $g\pi(\Delta_j)$ is a Du Val singular point.

By Claim(1), $\overline{F} := g_*F$ is a smooth ample Cartier divisor isomorphic to F (and also to E). Our relation (3.2.2) induces the relation (3.2.1). Thus, S_c satisfies the hypothesis in Lemma 3.2.

It is clear that $\pi : X \rightarrow S$ is the canonical resolution of the double covering $\pi_c : X_c \rightarrow S_c$ associated with the relation (3.2.1), because our contraction $g : S \rightarrow S_c$ here coincides with the map $g : S \rightarrow S_c (= T)$ constructed in Lemma 1.4. This proves Lemma 3.4.

Definition and Proposition 3.5. In Lemma 3.4, the double covering $\pi_c : X_c \rightarrow S_c$ is called *the canonical mapping model* of $\pi : X \rightarrow S$. This π_c is unique, because the map $g : S \rightarrow S_c$ is the contraction of all curves on S having zero intersection with π_*E and uniquely determined by π .

§4. Multiple-fiber free rational elliptic fibrations

Let S be a smooth rational surface with a multiple-fiber free relatively minimal elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$. Let F_1, F_2 be two smooth fibers of ψ . By the canonical divisor formula, one has $K_S \sim -F_1$. Hence one obtains the following relation:

$$(4.1.1) \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}(F_1 + F_2).$$

Let $\pi : X := \text{Spec } \bigoplus_{i=0}^1 \mathcal{O}(iK_S) \rightarrow S$ be the double covering associated with the relation (4.1.1). Then $\text{Gal}(X/S)$ is a cyclic group $\langle \sigma \rangle$ of order 2 such that $X/\sigma = S$.

The following lemma can be proved similarly as in Lemma 2.1.

Lemma 4.1. *Let S be a smooth rational surface with a multiple-fiber free relatively minimal elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$. Let $\pi : X \rightarrow S = X/\sigma$ be as above. Then the following two assertions hold true.*

(1) *One has $\pi^*F_i = 2E_i$ for an elliptic curve E_i isomorphic to F_i . The fixed locus X^σ is a disjoint union of E_1 and E_2 .*

(2) *X is a smooth K3 surface. The involution σ on X satisfies $\sigma^*\omega = -\omega$ for a non-zero holomorphic 2-form ω on X .*

The following Lemma 4.2 is the converse to Lemma 4.1.

Lemma 4.2. *Let (X, σ) be as in Theorem 1. Assume further that X^σ is a disjoint union of two elliptic curves E_1, E_2 . Then the pair (X, σ) can be realized in the way of Lemma 4.1, from a rational surface S satisfying all hypotheses there.*

Proof. By Lemma 1.11, $S = X/\sigma$ is a smooth rational surface and there exists a multiple-fiber free relatively minimal elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ with $F_i := \pi^*E_i = \pi^*(E_i)$ as smooth fibers. Now Lemma 4.2 follows from Lemma 1.2 (2).

§5. Rational elliptic fibrations with a multiple fiber

Let S be a smooth rational surface with a relatively minimal elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ such that $2F_2$ is the only multiple fiber of ψ , where F_2 is of Kodaira type I_s ($s \geq 0$).

By the canonical divisor formula, one has $K_S \sim -F_1 + F_2$ for a smooth fiber F_1 of ψ . Hence one obtains the following:

$$(5.1.1) \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}(F_1).$$

Let $\pi : X := \text{Spec} \bigoplus_{i=0}^1 \mathcal{O}(iK_S) \rightarrow S$ be the double covering associated with the relation (5.1.1). Then $\text{Gal}(X/S)$ is a cyclic group $\langle \sigma \rangle$ of order 2 such that $X/\sigma = S$.

The following Lemma 5.1 can be proved similarly as in Lemma 2.1.

Lemma 5.1. *Let S be a smooth rational surface with a relatively minimal elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ such that $2F_2$ is the only multiple fiber of ψ , where F_2 is of Kodaira type I_s ($s \geq 0$). Let $\pi : X \rightarrow S = X/\sigma$ be as above. Then the following two assertions hold true.*

- (1) *One has $\pi^*F_1 = 2E_1$ for an elliptic curve E_1 isomorphic to F_1 . The fixed locus X^σ is equal to E_1 .*
- (2) *X is a smooth K3 surface. The involution σ on X satisfies $\sigma^*\omega = -\omega$ for a non-zero holomorphic 2-form ω on X .*

The following Lemma 5.2 is the converse to Lemma 5.1.

Lemma 5.2. *Let (X, σ) be as in Theorem 1. Assume further that X^σ is a single elliptic curve E_1 . Then the pair (X, σ) can be realized in the way of Lemma 5.1, from a rational surface S satisfying all hypotheses there.*

Proof. By Lemma 1.11, there exists a fiber E_2 of $\varphi := \Phi_{|E_1|} : X \rightarrow \mathbf{P}^1$ such that E_1, E_2 are only σ -stable fibers of φ . Applying Lemma 1.5 to E_2 , we see that E_2 is of Kodaira type I_{2s} ($s \geq 0$).

By Lemmas 1.11 and 1.5, $S = X/\sigma$ is a smooth rational surface with a relatively minimal elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ such that $F_1 := \pi_*E_1$ is a smooth fiber of ψ and $\pi_*E_2 = 2F_2$ is the only multiple fiber of ψ , where F_2 is of Kodaira type I_s . Now Lemma 5.2 follows from Lemma 1.2 (2). This completes the proof of Lemma 5.2.

§6. Multiple-fiber free rational elliptic fibrations with a fiber of Kodaira type II, III, IV or I_n

Let S_c be a smooth rational surface with a multiple-fiber free relatively minimal elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$. Suppose that ψ has a singular fiber \overline{F}_∞ of either one of the Kodaira types II, III, IV and I_{n_∞} ($n_\infty \geq 1$).

By the canonical divisor formula, One has $K_{S_c} \sim -\overline{F}_1$ for a smooth fiber \overline{F}_1 of ψ . Hence one obtains the following relation:

$$(6.1.1) \quad \mathcal{O}(-K_{S_c})^{\otimes 2} \cong \mathcal{O}(\overline{F}_1 + \overline{F}_\infty).$$

Let $g : S \rightarrow S_c$ be the composite of blowing-ups of intersections of \overline{F}_∞ and their infinitely near points so that $F_\infty := g^*\overline{F}_\infty$ fits respectively one of Cases (α) , (β) , (γ) and (δ) in Lemma 1.5. We shall use the notation $\text{Supp} F_\infty = \sum_{j=1}^{n_\infty} D_{\infty,j} + \sum_j H_{\infty,j}$ there. Here $\sum_j D_{\infty,j}$ is a disjoint union of n_∞ (-4) -curves $D_{\infty,j}$.

The relation (6.1.1) induces the following relation, where $F_1 := g^*\overline{F}_1$

$$(6.1.2) \quad \mathcal{O}(-K_S)^{\otimes 2} \cong \mathcal{O}(F_1 + \sum_{j=1}^{n_\infty} D_{\infty,j}).$$

Let $\pi_c : X_c := \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_{S_c}) \rightarrow S_c$ and $\pi : X := \text{Spec} \oplus_{i=0}^1 \mathcal{O}(iK_S) \rightarrow S$ be the double coverings associated with the relations (6.1.1) and (6.1.2), respectively. Then both $\text{Gal}(X_c/S_c)$ and $\text{Gal}(X/S)$ are isomorphic to a cyclic group $\langle \sigma \rangle$ of order 2 such that $X_c/\sigma = S_c$ and $X/\sigma = S$.

The following Lemma 6.1 can be proved similarly as in Lemma 2.4. In fact, the second part of the assertion(4) follows from the observation: the Picard number $\rho(S) = 10 - K_S^2 = 10 + n_\infty$ by (6.1.2).

Lemma 6.1. *Let S_c be a smooth rational surface with a multiple-fiber free relatively minimal elliptic fibration $\psi : S_c \rightarrow \mathbf{P}^1$. Assume further that ψ has a singular fiber \overline{F}_∞ of Kodaira type II, III, IV or I_{n_∞} ($n_\infty \geq 1$). Let $\pi_c : X_c \rightarrow S_c = X_c/\sigma$ and $\pi : X \rightarrow S = X/\sigma$ be as above. Then the following four assertions hold true.*

(1) *One has $\pi^*F_1 = 2E_1$ for an elliptic curve E_1 isomorphic to F_1 , and $\pi^*D_{\infty,j} = 2C_{\infty,j}$ for a smooth rational curve $C_{\infty,j}$. The fixed locus X^σ is a disjoint union of E_1 , and n_∞ curves $C_{\infty,j}$ all contained in “the fiber” E_∞ (see (4) below). One has $n_\infty \leq 9$.*

(2) *X is a smooth K3 surface. The involution σ on X satisfies $\sigma^*\omega = -\omega$ for a non-zero holomorphic 2-form ω on X .*

(3) *There exists a σ -equivariant birational morphism $f : X \rightarrow X_c$ which induces the following commutative diagram with f as the minimal resolution and g a resolution*

$$\begin{array}{ccc} X & \xrightarrow{f} & X_c \\ \pi \downarrow & & \downarrow \pi_c \\ S & \xrightarrow{g} & S_c. \end{array}$$

(4) *One has $\pi^*F_\infty = 2E_\infty$, where E_∞ fits one of Cases (A), (B), (C) and (D) in Lemma 1.5 according to the type of F_∞ . One has also $\rho(X) \geq \rho(S) = 10 + n_\infty$.*

Definition 6.2. In Lemma 6.1, the double covering $\pi : X \rightarrow S$ is called *the canonical resolution* of $\pi_c : X_c \rightarrow S_c$.

Lemma 6.3. *Let (X, σ) be as in Theorem 1. Assume further that X^σ is a disjoint union of an elliptic curve E_1 and n ($n \geq 1$) smooth rational curves.*

Then the pair (X, σ) can be realized in the way of Lemma 6.1, from a rational surface S_c satisfying all hypotheses there, i.e., the quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution of the double covering $\pi_c : X_c \rightarrow S_c$ there.

Proof. By Lemma 1.11, $\varphi := \Phi_{|E_1|} : X \rightarrow \mathbf{P}^1$ has exactly two σ -stable fibers E_1, E_∞ . Applying Lemma 1.5, E_∞ fits one of Cases (A), (B), (C) and (D) there with $n_\infty = n$, and $F_\infty := \pi_* E_\infty$ fits respectively one of Cases (α) , (β) , (γ) and (δ) there. We use the notation $\text{Supp} E_\infty = \sum_{j=1}^n C_{\infty,j} + \sum_j G_{\infty,j}$ and $\text{Supp} F_\infty = \sum_{j=1}^n D_{\infty,j} + \sum_j H_{\infty,j}$ there, where $\pi^* D_{\infty,j} = 2C_{\infty,j}$. Now Lemmas 1.11 and 1.5 imply that S is a smooth rational surface, which has a multiple-fiber free elliptic fibration $\psi : S \rightarrow \mathbf{P}^1$ with $F_1 := \pi_*(E_1) = \pi(E_1)$ as its smooth fiber.

Let $g : S \rightarrow S_c$ be the smooth blowing-down of curves in F_∞ so that $\bar{F}_\infty := g_* F_\infty$ is of Kodaira type II, III, IV or I_n , according to the type of E_∞ . Now $\psi : S \rightarrow \mathbf{P}^1$ induces a multiple-fiber free relatively minimal elliptic fibration (Lemmas 1.11), also denoted by $\psi : S_c \rightarrow \mathbf{P}^1$. This S_c satisfies all hypotheses of Lemma 6.1.

By Lemma 1.2 (2), our quotient morphism $\pi : X \rightarrow S = X/\sigma$ coincides with the double covering associated with the relation (6.1.2). Thus π is the canonical resolution of the double covering $\pi_c : X_c \rightarrow S_c$ associated with the relation (6.1.1), because our contraction $g : S \rightarrow S_c$ here coincides with the map $g : S \rightarrow S_c$ constructed at the beginning of §6. This proves Lemma 6.3.

Definition and Proposition 6.4. In Lemma 6.3, the double covering $\pi_c : X_c \rightarrow S_c$ is called *the canonical mapping model* of $\pi : X \rightarrow S$. This π_c is unique, because $\varphi : X \rightarrow \mathbf{P}^1$ is the only elliptic fibration such that X^σ is contained in fibers of φ , and there is a unique smooth blowing-down $g : S \rightarrow S_c$ of curves in F_∞ such that $g_* F_\infty$ is a minimal fiber.

§7. Proofs of Theorems 1, 3 and 4 and Corollary 5

Theorem 1 is a consequence of Lemmas 1.2, 1.11, 2.9, 3.4, 4.2, 5.2 and 6.3. Theorem 4 follows from Lemmas 2.1 and 2.9.

Next we prove Corollary 5 using Theorem 3'. The first part of (1) is proved in Lemma 2.1(4). By Remark 2.6 and Example 2.7, for any $1 \leq n \leq 10$, there is a K3 surface X with an involution σ such that X^σ is a disjoint union of n smooth rational curves C_i .

Let $\pi : X \rightarrow S := X/\sigma$ be the quotient morphism. As in Lemma 2.1 (4), let $v_3 : S \rightarrow S_3$ be the contraction of n (-4) -curves $D_i := \pi_* C_i$. Then S_3 is a rational log Enriques surface of index 2 with n cyclic-quotient singular points $v_3(D_i)$ of Brieskorn type $C_{4,1}$ as its only singular points (Lemma 1.4). This proves Corollary 5(1).

Now let R be a rational log Enriques surface of index 2 with 10 singular points r_i of Cartier index 2. Let $\pi : X \rightarrow S := X/\sigma$ be the canonical resolution of the canonical covering $\bar{\pi} : W \rightarrow R$ (Definition 2.2). By Lemma 2.1, one has $n = 10$ and

$n = \sum_{i=1}^{10} n_i \leq 10$. So, $n_i = 1$ for all i . Hence $Sing R$ consists of 10 cyclic-quotient singular points r_i of Brieskorn type $C_{4,1}$ and several Du Val singular points.

In the notation of Lemma 2.1, $D_i := q^{-1}(r_i)$ is a (-4) -curve on S and X^σ is the disjoint union of 10 smooth rational curves C_i where $\pi^* D_i = 2C_i$. By Theorem 3, (X, σ) is isomorphic to Shioda-Inose's unique pair. Now we have only to show that R has no Du Val singular points, for then $q : S = X/\sigma \rightarrow R$ is just the contraction of $\pi(X^\sigma)$ and Corollary 5(2) follows from the uniqueness of the pair (X, σ) .

Suppose to the contrary that R has a Du Val singular point r_0 . Then $q^{-1}(r_0)$ consists of (-2) -curves disjoint from $\sum_{i=1}^{10} D_i$. Hence $\pi^{-1}q^{-1}(r_0)$ consists of smooth rational curves disjoint from $\sum_{i=1}^{10} C_i = X^\sigma$. By the following Claim(1), each (-2) -curve on X is σ -stable and we reach a contradiction to Lemma 1.1 (3).

Claim(1). $\sigma^*|PicX = id$.

This is proved in [OZ1, Lemma 3.3]. Actually, in the notation of Lemma 2.1 (4), σ stabilizes 19 curves $\sum_{i=1}^{10} C_i + \sum_{j=1}^9 G_j$ as well as the pull back of the generator of $PicX_2$.

This proves Claim(1) and also Corollary 5.

Now we prove Theorem 3' in §1 which is stronger than Theorem 3. The inequality $0 \leq m \leq 10$ follows from Theorem 1. One sees that m can attain any value in this range by Theorem 1(1) and the proof of Corollary 5(1). Assume that $m = 10$. Then, by Theorem 1, the quotient morphism $\pi : X \rightarrow S := X/\sigma$ is the canonical resolution of its canonical mapping model $\pi_c : X_c \rightarrow S_c$ given in either Lemma 2.4 with $n = 10$, or Lemma 3.2 with $n = 9$, or Lemma 6.1 with $n_\infty = 9$ and \overline{F}_∞ of Kodaira type I_9 . Now Theorem 3'(1) follows.

We now prove Theorem 3'(2). Then π_c is given in Lemma 2.4 with $n = 10$. By Lemma 2.3, our $\pi : X \rightarrow S = X/\sigma$ is the canonical resolution of the canonical double covering $\overline{\pi} : W \rightarrow R$ of a rational log Enriques surface R of Type A_{19} . This R determines uniquely the pair (X, σ) (see Lemma 2.1). Now Theorem 3'(2) follows from [OZ1, Theorem 2] saying that there is, up to isomorphisms, only one rational log Enriques surface of Type A_{19} .

We need the following Lemma 7 to prove Theorem 3' (3)(4).

Lemma 7. *Let S_{ci} ($i = 1, 2$) be a smooth rational surface with a multiple-fiber free relatively minimal elliptic fibration $\psi_i : S_{ci} \rightarrow \mathbf{P}^1$ which has exactly one singular fiber \overline{F}_{ci} of Kodaira type I_9 and three singular fibers of Kodaira type I_1 as its only singular fibers. Let M_i be a (-1) -curve on S_{ci} .*

Then there is an isomorphism $\tau : S_{c1} \rightarrow S_{c2}$ such that $\tau^\overline{F}_{c2} = \overline{F}_{c1}$ and $\tau^*M_2 = M_1$.*

Finally, there exists a smooth rational curve H_0 on S_{c1} such that $H_0^2 = 0$ and H_0 is a 2-section of ψ_1 passing through an intersection of \overline{F}_{c1} .

Proof. By the hypothesis on ψ_i and the canonical divisor formula, one has $K_{S_{ci}} + \overline{F}_{ci} \sim 0$. Hence M_i is a cross-section. Thus we can write $\overline{F}_{ci} = \sum_{j=1}^9 D_{i,j}$ so that $M_i \cdot D_{i,j} = 1$ where $D_{i,j} \sim -D_{i,1} - \dots - D_{i,j-1}$. Let $h : S_{c1} \rightarrow \mathbf{P}^1$ be

the smooth blowing down of $M_i + \sum_{j=1}^8 D_{i,j}$ into the node d_i of the nodal cubic $h_i(D_{i,9})$.

We may assume that both $h_i(D_{i,9})$ are equal to $\overline{D} : Y^2Z - X^2(X+Z) = 0$ in \mathbf{P}^2 . Note that the projective transformation $\tau_1 : \mathbf{P}^2 \rightarrow \mathbf{P}^2$, where $\tau_1(X) = -X$, $\tau(Y) = Y$, $\tau(Z) = -Z$, stabilizes \overline{D} and switches two local irreducible components of \overline{D} at its node $d := [0 : 0 : 1]$. Note also that $\Lambda_i := h_{i*}|\overline{F}_{ci}|$ is a 1-dimensional linear system satisfying the following hypothesis(*):

(*) Λ_i contains the nodal cubic $\overline{D} : Y^2Z - X^2(X+Z) = 0$ as a member. A general member of Λ_i is an elliptic curve and touches, with order 8, the local irreducible component of \overline{D} , tangent to $Y + \varepsilon_i X = 0$, at the node d . Here $\varepsilon_i = \pm 1$.

Claim(2). For a given ε_i , the Λ_i above is the only 1-dimensional linear system satisfying the hypothesis(*) above.

Indeed, suppose that G_1, G_2 are two distinct elliptic curves, each of which touches, with order 8, the local irreducible component of \overline{D} , tangent to $Y + \varepsilon X = 0$, at the node d . Let $h : W \rightarrow \mathbf{P}^2$ be the blowing-up of d and 7 of its infinitely near points, such that \overline{D} and the proper transform G'_i on W of G_i have no intersection. It is easy to see that the pull-back on W of \overline{D} is a simple loop of 8 (-2) -curves and one (-1) -curve which together generate $(\text{Pic}W) \otimes \mathbf{Q}$, whence one can check that G'_1, G'_2 are numerically (and hence also linearly) equivalent. Therefore, $G_1 \sim G_2$. This proves Claim(1).

There is a projective transformation $\tau : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $\tau^*\Lambda_2 = \Lambda_1$. Indeed, let $\tau = id$ if $\varepsilon_1 = \varepsilon_2$, and let $\tau = \tau_1$ above otherwise. This τ induces an isomorphism between S_{c1} and S_{c2} required by Lemma 7, because the linear system $|\overline{F}_i|$ is just the unique minimal resolution of base points (i.e., d_i and its infinitely near points) in Λ_i and hence uniquely determined by Λ_i .

For the last paragraph of Lemma 7, we take the projective line \overline{H}_0 through d_1 and tangent to the local irreducible component, other than the one in the hypothesis(*), of $h_1(D_{1,9})$ at its node d_1 . Then the proper transform $H_0 := h'_1\overline{H}_0$ is through $D_{1,8} \cap D_{1,9}$ and the one required by Lemma 7. This completes the proof of Lemma 7.

As a consequence to Lemma 7, one obtains:

Corollary 8. *There is, up to isomorphisms, only one log del Pezzo surface T with a type A_8 Du Val singular point as its only singular point.*

Proof. Let T_i be two surfaces both satisfying all hypotheses of Corollary 8. Let $v_i : T'_i \rightarrow T_i$ be the minimal resolution. Then $\Delta_i := v_i^{-1}(\text{Sing}T_i)$ has Dynkin type A_8 . Note that $-K_{T'_i} = -v_i^*K_{T_i}$ is nef and big. Hence $9 \geq 10 - (K_{T'_i})^2 = \rho(T'_i) = 8 + \rho(T_i) \geq 9$. So $(K_{T'_i})^2 = 1$ and the Picard number $\rho(T_i) = 1$.

The Riemann-Roch theorem and the vanishing theorem [KMM, Theorem 1.2.2]

imply that $\dim | -K_{T'_i}| = 1$. Thus, $Bs| -K_{T'_i}|$ consists of a single point t_i and a general member F'_i of $| -K_{T'_i}|$ is an elliptic curve (see also [D, Theorem 1, p. 39]).

Let $w_i : S_{ci} \rightarrow T'_i$ be the blowing-up of t_i . Then there exists a multiple-fiber free relatively minimal elliptic fibration $\psi_i : S_{ci} \rightarrow \mathbf{P}^1$ with $w'_i F'_i$ as a general fiber and the (-1) -curve $M_i := w_i^{-1}(t_i)$ as a cross-section.

It is easy to see that the singular fiber \overline{F}_{ci} of ψ_i containing $w'_i \Delta_i$ is of Kodaira type I_9 . By [P, the list], our ψ satisfies all hypotheses in Lemma 7. So, there exists an isomorphism $\tau : S_{c1} \rightarrow S_{c2}$ such that $\tau^* M_2 = M_1$ and $\tau^* \overline{F}_{c2} = \overline{F}_{c1}$. This τ induces an isomorphism between T_1 and T_2 . This proves Corollary 8.

We now continue the proof of Theorem 3' (3)(4).

Let (S_c, ψ) be the unique pair, modulo isomorphisms, in Lemma 7. Set $\overline{F}_s := \psi^{-1}(s)$. We may assume that $\overline{F}_\infty, \overline{F}_0, \overline{F}_1, \overline{F}_{s_0}$ are respectively of Kodaira types I_9, I_1, I_1, I_1 . Write $\overline{F}_\infty = \sum_{i=1}^9 \overline{D}_i$ so that $\overline{D}_i \cdot \overline{D}_{i+1} = 1$, where $\overline{D}_{9+i} := \overline{D}_i$.

As in Lemma 6.1, let $g_{ell} : S_{ell} \rightarrow S_c$ be the smooth blowing-up of the 9 intersections in \overline{F}_∞ so that $F_\infty := g_{ell}^* \overline{F}_\infty = D_1 + 2H_1 + D_2 + 2H_2 + \cdots + D_9 + 2H_9$ with $D_i := g'_{ell} \overline{D}_i$, as given in Lemma 1.5 Case(δ). ψ on S_c induces an elliptic fibration, also denoted by $\psi : S_{ell} \rightarrow \mathbf{P}^1$ with $F_s := g_{ell}^* \overline{F}_s$ as a fiber. Now a relation identical to (6.1.1) (but with different labelling) induces the following relation identical to (6.1.2) except the labelling :

$$(3.2) \quad \mathcal{O}(-K_{S_{ell}})^{\otimes 2} \cong \mathcal{O}(F_s + \sum_{i=1}^9 D_i).$$

As in Lemma 6.1, let $\pi : X_s \rightarrow S_{ell} = X_s/\sigma_s$ be the double covering associated with the relation (3.2). Then $X_s/\sigma_s = S_{ell}$ is independent of the choice of s and Theorem 3'(3i) is true.

When F_s is smooth, the pair (X_s, σ_s) fits Lemma 6.1 with $n_\infty = 9$ and is of Type(Ell). Now Theorem 3'(iv) follows from Theorem 1 and Lemmas 6.1, 6.3 and 7.

When $s = \infty$, the right hand side of the relation (3.2) is divisible by 2 in $Pic S_{ell}$. So Theorem 3'(3ii) is true.

Suppose that $s = 0, 1$, or s_0 . By the construction in Lemma 2.4 or Example 2.8, there is a smooth blowing-up $X/\sigma \rightarrow X_s/\sigma_s$ of the node of F_s , where (X, σ) is Shioda-Inose's unique pair, constructed also in Example 2.8 with $(n_1, n_2 = n_\infty) = (1, 9)$. Thus Theorem 3'(3iii) is true. This proves Theorem 3'(3).

By Lemma 7, there is a smooth rational curve \overline{H}_0 on S_c such that $\overline{H}_0^2 = 0$ and that \overline{H}_0 is a 2-section of ψ through the intersection $\overline{D}_9 \cap \overline{D}_1$ after rotating the indices. Set $H_0 := g_{ell}^{-1}(\overline{H}_0)$, which is a (-1) -curve. For each of $s = 0, 1, s_0$, H_0 intersects the fiber F_s at its two distinct smooth points, for otherwise the inverse on X of H_0 is a union of a (-2) -curve and its σ -conjugate and we reach a contradiction to the fact that $\sigma^*|Pic X = id$ (see Claim(1) in the proof of Corollary 5). Now since H_0 is a 2-section of ψ , H_0 has a contact of order 2 with a smooth fiber F_{s_1} say, and intersects each fiber F_s ($s \neq \infty, s_1$) at its two smooth points.

Let $S_{ell} \rightarrow S_{gn2}$ be the smooth blowing-down of H_0 . Then (3.2) induces the following relation, where F'_s, D'_i and also H'_i (for later use) are images of F_s, D_i, H_i

$$(3.3) \quad \mathcal{O}(-K_{S_{gn2}})^{\otimes 2} \cong \mathcal{O}(F'_s + \sum_{i=1}^9 D'_i).$$

Note that F'_∞ is a non-reduced simple loop.

Let $g_{gn2} : S_{gn2} \rightarrow \overline{S}_{gn2}$ be the contraction of the linear chain $D'_1 + H'_1 + D'_2 + H'_2 + \cdots + D'_9$ into a cyclic quotient singularity of Brieskorn type $C_{36,17}$. Then one obtains:

$$(3.4) \quad \mathcal{O}(-K_{\overline{S}_{gn2}})^{\otimes 2} \cong \mathcal{O}(g_{gn2*}F'_s).$$

Note that $g_{gn2*}F'_\infty$ is twice of a nodal rational curve with the only singular point of \overline{S}_{gn2} as its node, and $g_{gn2*}F'_s$ ($s \in \mathbf{P}^1 - \{\infty\}$) is a curve of arithmetic genus 2 and away from the singular point of \overline{S}_{gn2} ; moreover, for each of $s = 0, 1, s_0$ the curve $g_{gn2*}F'_s$ is rational with two simple nodes, $g_{gn2*}F'_{s_1}$ is elliptic with a simple cusp, and for each $s \neq \infty, 0, 1, s_0, s_1$ the curve $g_{gn2*}F'_s$ is elliptic with a simple node.

Since the Picard number $\rho(\overline{S}_{gn2}) = \rho(S_{gn2}) - 17 = \rho(S_{ell}) - 18 = \rho(S_c) + 9 - 18 = 1$, our \overline{S}_{gn2} is the unique log del Pezzo surface of Cartier index 2 and Picard number 1 (see Lemma 3.2(4)).

By Lemmas 3.1 and 3.2(4), $\dim |-2K_{\overline{S}_{gn2}}| = 3$. So $\mathbf{P}(|-2K_{\overline{S}_{gn2}}|) = \mathbf{P}(|g_{gn2}^*(-2K_{\overline{S}_{gn2}})|) = \mathbf{P}^3$. Let B denote the 1-dimensional linear subsystem of $|-2K_{\overline{S}_{gn2}}|$ consisting of $g_{gn2*}F'_s$, the direct image of $F_s = g_{ell}^*\overline{F}_s$ ($s \in \mathbf{P}^1$).

By the constructions, our $g_{gn2} : S_{gn2} \rightarrow \overline{S}_{gn2}$ coincides with the $g : S \rightarrow S_c$ in Lemma 1.4 or Lemma 3.2 with $n = 9$. As in Lemma 3.2, for any member $F'_t \in |g_{gn2}^*(-2K_{S_c})|$ ($t \in \mathbf{P}^3$), we let $\pi : Y_t \rightarrow S_{gn2} = Y_t/\sigma_t$ be the double covering associated with the relation (3.3) where F'_s is replaced by F'_t . Then $Y_t/\sigma_t = S_{gn2}$ is independent of the choice of t and Theorem 3'(4i) is true.

When F'_t ($t \in \mathbf{P}^3$) is smooth, $\pi : Y_t \rightarrow S_{gn2} = Y_t/\sigma_t$ coincides with $\pi : X \rightarrow S = X/\sigma$ given in Lemma 3.2 with $n = 9$, and (Y_t, σ_t) is of Type(Gn2). Conversely, by Theorem 1 and Lemmas 3.2 and Lemma 3.4, every pair of Type(Gn2) is isomorphic to (Y_t, σ_t) for some $t \in \mathbf{P}^3$. So Theorem 3'(4ii) is true.

Theorem 3'(4iii) follows from the construction of $S_{ell} \rightarrow S_{gn2}$ and the definition of B above. This completes the proof of Theorem 3'.

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